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Some Two-on-Two Homogeneous Stochastic Combats*

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In this article we consider two versions of two-on-two homogeneous stochastic combat and develop expressions, in each case, for the state probabilities. The models are natural generalizations of the exponential Lanchester square law model. In the first version, a marksman whose target is killed resumes afresh the killing process on a surviving target; in the second version, the marksman whose target is killed merely uses up his remaining time to a kill on a surviving target. Using the state probabilities we then compute such important combat measures as (1) the mean and variance of the number of survivors as they vary with time for each of the sides, (2) the win probabilities for each of the sides, and (3) the mean and variance of the battle duration time. As an application, computations were made for the specific case of a gamma (2) interfiring time random variable for each side and the above combat measures were compared with the appropriate exponential and deterministic Lanchester square law approximations. The latter two are shown to be very poor approximations in this case.

1. INTRODUCTION

The principal motivation for this work is the development of more realistic small-to-moderate-size firefight models. It is an extension of the work started in references [1] and [7] which treat the one-on-one and homogeneous two-on-one stochastic combat models, respectively.

The overall framework within which all these works lie is described in reference [3] where (1) a comprehensive examination is presented of the nature of combat and the status of corresponding theory, and (2) a proposal toward a theory of combat is set forth. The basic conclusion reached in reference [3] is that current modeling, no matter how "realistic" it is claimed to be, is deficient and not based on any firmly established theory. One only needs to look at an application of the classical Lanchester square law to large numbers of opposing forces (inserted as initial numbers) to see the following two egregious assumptions, concerning the nature of combat:

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1. Combat is homogeneous. This in effect assumes that in 1000-on-1000 battles all the forces are simultaneously engaging the enemy in the same way. Common sense dictates that the effects of cover, concealment, weather, terrain obstacles, terrain corridors, the effective range of different weapons, countless other environmental factors, and tactical deployment force the opponents into many smaller firefights. An excellent field study in this connection appears in reference [9] where it was found that a structured relationship for the minibattles may be constructed showing how some battles occur simultaneously (or in parallel) while others involve the participants in a series of battles through time.

2. Combat is deterministic. To believe that combat is not stochastic would require that if a firefight could be repeated under exactly the same conditions, every event that occurred in the first replication would reoccur on, say, the second replication in exactly the same order at exactly the same time. This truly would be incredible. Stochasticity in combat is indeed significant and an extensive discussion of it may be found in reference [2]. In addition, reference [9] describes an excellent experiment showing not only stochasticity in a combat process but also its considerable variance.

We hold to the view described in reference [3] that the ultimate development of realistic combat models involving large numbers of weapons will depend on successfully modeling (1) the decomposition of the large battle into the separate small engagements (in this connection see references [4] and [5]), and (2) the attrition process in the separate engagements. This study is directed toward (2), the realistic modeling of small-to-moderate-size engagements.

Furthermore, we believe that to achieve realism in the attrition process we must proceed, as it has generally been done in the physical sciences, from the simple to the more complex, and so this study is the obvious successor model to those described in references [1] and [7]. It provides what we believe to be substantially more realism in the extant two-on-two stochastic (more accurately, exponential) Lanchester square law combat models. It does so by removing the extremely simplifying assumption that the interkilling time random variable is the same negative exponential distribution (NED) from kill to kill (this is a consequence of assuming that both the single-shot kill probability and the NED interfiring time random variable are the same from round to round). Allowing both the single-shot kill probability and interfiring time random variable to vary from round to round, of course, complicates the analysis substantially and numerical techniques are required to produce state probabilities in any specific case. We do, however, retain the homogeneity assumption that all combatants on a side possess identical characteristics (but which are not necessarily the same for both sides).

The two-on-two raises the question of how a marksman whose original target is killed by another handles a surviving target. In this article we consider two versions of how this situation is dealt with. In the first version the marksman starts the killing process all over again on the surviving target, and in the second version the marksman uses up the remaining time to a firing (or a killing) on the surviving target. In the exponential Lanchester case there is no difference in these two versions because of the no-memory property

associated with the exponential random variable. *In each of these versions we have allowed all possible values for a side's breakpoint* (the number of survivors at the time the side loses).

In the following sections the problem is precisely formulated for both versions and the state probabilities are then derived. These state probabilities, which are functions of time, contain all the pertinent information about this nonstationary stochastic process, and, in fact, in terms of them we can write expressions for the following important combat measures of effectiveness:

1. The mean and variance of the number of survivors as they vary with time for each of the sides. At $t = \infty$, this gives the mean and variance of the number of survivors of the battle for each of the sides.
2. The win probabilities for each of the sides.
3. The mean and variance of the battle duration time T_D. Actually one may write the distribution function of battle duration $F_{T_D}(t)$, in terms of the state probabilities and the equivalent probability density function $f_{T_D}(t)$ in terms of their derivatives.

The technique used in this article results in n-fold iterated integrals for the state probabilities. The integrands are complex products of the density and complementary distribution functions of the interkilling time random variables of both sides. The dimension of the n-fold integrals, for any particular state probability, is equal to the number of kills corresponding to the state. Thus, e.g., in the zero-breakpoint-for-both-sides case we get up to three-dimensional iterated integrals; and the computation time to get all state probability functions plus eight overall battle parameters is from two to three hours on a Sperry 1100/82 when the interfiring time random variables are gamma(2) for both sides.

We also note here that the present method certainly can be extended to the three-on-two and even the heterogeneous versions of battles up to that size, where by heterogeneity we mean that the characteristics of the combatants on a side are allowed to be different; and these extensions are presently under way. Desirable as it is to have exact solutions, in view of the computer times involved in using the present approach, we feel that alternative methods must be considered. These include the following:

1. Simulation. We have already developed arbitrary a_0-on-b_0 versions of these models, including the requisite statistical techniques for their use. We are presently in the evaluation process.

2. Approximations. We have defined a nonhomogeneous Poisson process approximation which results in vastly simpler analytical expressions for all the parameters of interest and for which the computer time required to produce answers in any specific case is substantially less than required by our exact model. Furthermore, the simulation version of this approximation is much faster than the exact version. We are presently evaluating this approximation.

3. Other exact techniques. These may result in simpler expressions for the state probability functions than we have developed and, therefore, presumably require much less computer time in any specific application. As of this time we have not come up with any such technique nor have we been successful in simplifying our results.

2. THE MODELS

Two sides, A and B, conduct a continuous engagement satisfying the following essentially Lanchester square law assumptions:

1. There are initially two on each side.
2. Every member of A side picks a B opponent at random (all are visible and in range).
3. Each marksman fires until killed or until his target is killed and resumes firing immediately on the survivor in one of two distinct ways which are described below and denoted as Versions 1 and 2.
4. The *interkilling* time random variable does not change from kill to kill and is identical for all members of the A side.
5. All fire independently.
6. The ammunition supply is unlimited.
7. Simular assumptions apply to the B side.
8. The battle continues until one side reaches its breakpoint (the number on a side at the time it loses).

The two modes of resuming firing on a survivor are as follows.

Version 1. Consider a given marksman firing at a target. Whether his target is killed by him or the other member of his side, he resumes afresh the interkilling process on the survivor.

Version 2. If the marksman's target is killed by him, he starts afresh the interkilling process on the survivor. If, on the other hand, his target is killed by the other member on the side, his remaining time to a firing (or a killing) is carried over to the survivor. The jargon we use to describe Versions 1 and 2 are "reselect on" and "reselect off," respectively.

When we consider all the possible breakpoints for sides A and B, respectively, we get a total of five models. These are shown in Table 1 below with a model numbering scheme for ease of reference.

3. GENERAL SOLUTIONS

Our solution technique will depend on knowing the interkilling time random variable's density and complementary distribution functions for each side. So long as they can be described analytically or developed in tabular form (with exact entries using some numerical techniques or estimated using Monte Carlo) the formulas derived in this article may be used to compute the state

Table 1. A brief description of the five models.

Model no.	Breakpoints Side A	Side B	"Reselect"
1.1	0	0	on
1.2	0	0	off
2.1	1	0	on
2.2	1	0	off
3	1	1	Not material

probabilities and subsequently overall combat and time-varying characteristics (this means, for example, that the models handle variable single-shot kill probabilities and, for that matter, variable interfiring time random variables). However, questions regarding ammunition consumption cannot be addressed because our solution technique perforce loses all information on number of rounds fired.

In this article the solutions to Models 1.1 and 1.2 are given in some detail. These two have the largest number of states and are the most difficult to treat. In particular, we begin this section with Model 1.1 and proceed far enough along to give the reader some notion of how state probabilities are derived and then place the remainder of the derivations for Model 1.1 and all of Model 1.2 in Appendix 1. There we present also only the results for the other breakpoint combinations (three more models).

Notation we use throughout the remainder of this article is as follows:

a_0 = the initial number on side A (at time 0)
a_f = breakpoint for side A, i.e., the number on side A at the time the A side loses (breaks and runs)
b_0 = the initial number on side B (at time 0)
b_f = breakpoint for side B, i.e., the number on side B at the time the B side loses (breaks and runs)
$f_A(t)$, $G_A(t)$ = density function and complementary distribution function for the time-to-kill-of-a-passive-target random variable, side A
$f_B(t)$, $G_B(t)$ = density function and complementary distribution function for the time-to-kill-of-a-passive-target random variable, side B
$A(t)$ = random variable, number alive on side A at time t
$B(t)$ = random variable, number alive on side B at time t
$p_{ab}(t) = P[A(t) = a, B(t) = b]$, a state probability function
$m_A(t) = E[A(t)]$, expected value of $A(t)$
$m_B(t) = E[B(t)]$, expected value of $B(t)$
$\sigma_A(t)$ = standard deviation of $A(t)$
$\sigma_B(t)$ = standard deviation of $B(t)$
$P[i]$ = probability i side wins, $i = A, B$
T_D = random variable, time duration of combat
$G_{T_D}(t)$ = complementary distribution function for T_D
μ_{T_D} = expected value of T_D
σ_{T_D} = standard deviation of T_D
ν_i = mean interkilling time on side i, $i = A, B$
$r_A = 1/\nu_A$ = A's kill rate (attrition coefficient for side B),
$r_B = 1/\nu_B$ = B's kill rate (attrition coefficient for side A)

and whenever the single-shot kill probability and interfiring time random variables are the same from round to round we use the notation

p_i = the constant kill probability of all constants on side i, $i = A, B$
μ_i = mean interfiring time on side i, $i = A, B$
$r_A = 1/\nu_A = p_A/\mu_A$
$r_B = 1/\nu_B = p_B/\mu_B$.

In some of our calculations we use the backward recurrence time technique to write the state probability equations. If at time t we define Y to be the time since the last event (kill), then the first-order probability that an A marksman

will kill in the interval $(t, t+\Delta)$ is given by

$$r_A(y)\Delta = f_A(y)/G_A(y)\Delta,$$

and

$$r_B(y)\Delta = f_B(y)/G_B(y)\Delta$$

for a B marksman. The $r(y)$s are the instantaneous kill rates for each marksman. See [1] for a discussion of the backward recurrence time technique.

Model 1.1 ($a_0 = 2$, $a_f = 0$; $b_0 = 2$, $b_f = 0$; "Reselect On")

The analyses for both Models 1.1 and 1.2 proceed in the order shown in Figure 1; namely, we will write an expression for $p_{22}(t)$, then $p_{12}(t)$ and $p_{21}(t)$, etc. In many of our considerations it becomes convenient to delineate the various aiming configurations. Because of the homogeneity of the combatants on each side the total number of distinct configurations is four, which is the product of the two ways the As aim at the Bs (both As aim at the same B or at different Bs) and the two ways the Bs aim at the As (both Bs aim at the same A or at different As). The analysis for Model 1.1 is carried out in terms of the initial aiming configurations. Thus what we do is break up each of the states shown in Figure 1, except (2, 2), into subsets that are associated with the initial aiming configuration. We define these subsets now and the associated state probabilities.

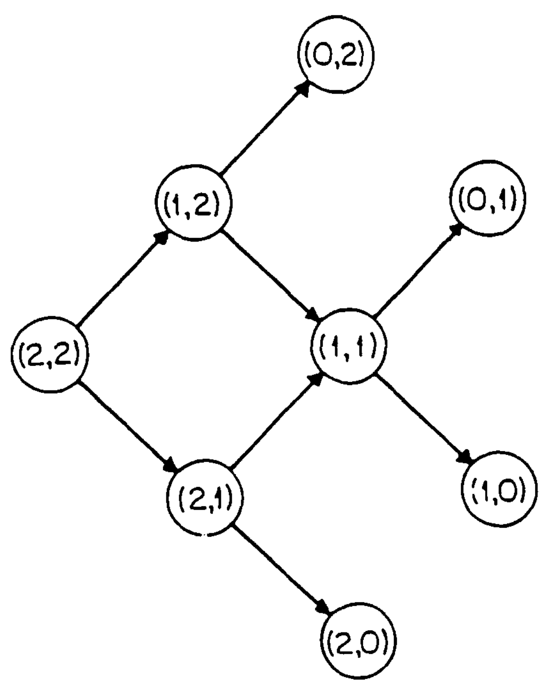

Figure 1. Sequence of states in the two-on-two combat with zero breakpoints.

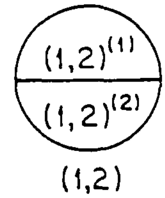

Figure 2. Decomposition of state $(1, 2)$.

Let us first consider the state $(1, 2)$ and break it up into states $(1, 2)^{(1)}$ and $(1, 2)^{(2)}$ as shown in Figure 2. State $(1, 2)^{(1)}$ is the state $(1, 2)$ achieved from the initial aiming configuration in which both Bs are aiming at the same A and $(1, 2)^{(2)}$ is the state $(1, 2)$ achieved from an initial aiming configuration in which each B is aiming at a different A. The corresponding

$$p_{12}(t) = p_{12}^{(1)}(t) + p_{12}^{(2)}(t).$$

Similarly,

$$p_{21}(t) = p_{21}^{(3)}(t) + p_{21}^{(4)}(t),$$

where the state $(2, 1)^{(3)}$ is state $(2, 1)$ achieved from an initial aiming configuration in which both As are aiming at the same B and state $(2, 1)^{(4)}$ is state $(2, 1)$ achieved from an initial aiming configuration in which each A is aiming at a *different* B. The reason for naming these states as $(2, 1)^{(3)}$ and $(2, 1)^{(4)}$ instead of $(2, 1)^{(1)}$ and $(2, 1)^{(2)}$, respectively, will become clear when we write down the decomposition for the states $(1, 1)$, $(0, 1)$, and $(1, 0)$.

As far as states $(0, 2)$ and $(2, 0)$ are concerned, they may be decomposed into the states shown in Figure 3, in which we have that

$$p_{02}(t) = p_{02}^{(1)}(t) + p_{02}^{(2)}(t),$$

and

$$p_{20}(t) = p_{20}^{(3)}(t) + p_{20}^{(4)}(t).$$

As above, the superscripts (1) and (2) correspond to both Bs aiming at the same A and each B aiming at a different A, respectively; whereas, the superscripts (3) and (4) correspond to both As aiming at the same B and each A aiming at a different B, respectively.

Figure 3. Decomposition of states $(0, 2)$ and $(2, 0)$.

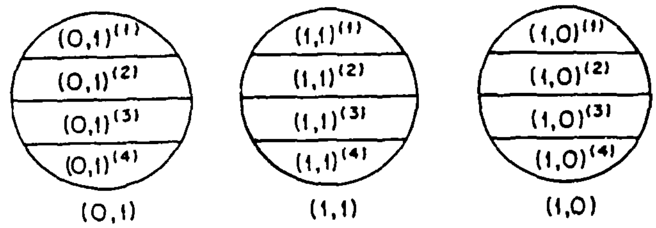

Figure 4. Decomposition of states (0, 1), (1, 1), and (1, 0).

We finally decompose the states (1, 1), (0, 1), and (1, 0) into the states shown in Figure 4. In thinking of these states the reader should clearly keep in mind the meaning of the superscripts (i), $i = 1, 2, 3, 4$. They are

(1) Initially both Bs are aiming at the same A and the first kill is of an A by a B.
(2) Initially each B is aiming at a different A and the first kill is of an A by a B.
(3) Initially both As are aiming at the same B and the first kill is of a B by an A.
(4) Initially each A is aiming at a different B and the first kill is of a B by an A.

Bearing this in mind, $(0, 1)^{(1)}$, say, is a state resulting from a B achieving the first kill, followed by the surviving A killing a B, and finally the surviving B killing the surviving A, and all this from an initial aiming configuration in which both Bs were aiming at the same A.

1. $p_{22}(t)$. We begin by setting down immediately

$$p_{22}(t) = (G_A(t))^2(G_B(t))^2. \tag{1}$$

which merely states that each of the contestants has a time to kill $> t$.

2. $p_{12}(t)$, $p_{21}(t)$. To compute $p_{12}(t)$ consider Figure 5 below, which shows that an A is killed by a B in the time interval $(t - \eta - d\eta, t - \eta)$ with no subsequent killings until beyond t. Now either both Bs were aiming at the same A or they were not. Each of these initial aiming configurations has probability of $\frac{1}{2}$. We now define $\overset{(1)}{p}_{12}(t, \eta)\, d\eta$ = probability that both Bs are aiming at the same A, one of the Bs kills an A in $(t - \eta - d\eta, t - \eta)$, and there are no other killings until beyond t, and $\overset{(2)}{p}_{12}(t, \eta)\, d\eta$ = probability that each B is aiming at a different A, one of the Bs kills an A in $(t - \eta - d\eta, t - \eta)$, and there are no other killings until beyond t.

Once we write expressions for $\overset{(1)}{p}_{12}(t, \eta)$ and $\overset{(2)}{p}_{12}(t, \eta)$ we can get $p_{12}(t)$ as

$$p_{12}(t) = \overset{(1)}{p}_{12}(t) + \overset{(2)}{p}_{12}(t) = \int_0^t d\eta\, (\overset{(1)}{p}_{12}(t, \eta) + \overset{(2)}{p}_{12}(t, \eta)).$$

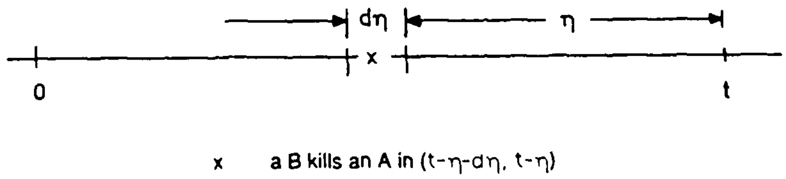

Figure 5. Definition of the variable η for the computation of $p_{12}(t)$.

Let us first consider $p_{12}^{(1)}(t, \eta)\, d\eta$. It can be written, because of the independence assumption of all the firers, as $\frac{1}{2}$ times the product of

$2f_B(t-\eta)\, d\eta$ = probability that one of the two Bs kills an A in the time interval $(t-\eta-d\eta, t-\eta)$
$G_B(t-\eta)$ = probability that nonkilling B has a kill time $> t-\eta$
$(G_B(\eta))^2$ = probability both Bs start over again on surviving A at time $t-\eta$ and have a kill time $> \eta$
$G_A(t-\eta)$ = probability that killed A had a time to kill $> t-\eta$.

and finally $G_A(t)$ = probability that the surviving A has a time to kill $> t$. Thus

$$p_{12}^{(1)}(t, \eta) = (1/2)[2f_B(t-\eta)G_B(t-\eta)(G_B(\eta))^2 G_A(t-\eta)G_A(t)].$$

Similarly,

$$p_{12}^{(2)}(t, \eta) = (1/2)[2f_B(t-\eta)G_B(\eta)G_B(t)G_A(t-\eta)G_A(t)],$$

where

$2f_B(t-\eta)\, d\eta$ = probability one of the two Bs kills an A in the time interval $(t-\eta-d\eta, t-\eta)$
$G_B(\eta)$ = probability that the killing B starts the firing process all over on surviving A and has a time to kill $> \eta$
$G_B(t)$ = probability nonkilling B has a time to kill $> t$
$G_A(t-\eta)$ = probability that killed A had a time to kill $> t-\eta$,

and finally $G_A(t)$ = probability that the surviving A has a time to kill $> t$. Thus we get

$$p_{12}(t) = G_A(t)\int_0^t d\eta\, f_B(t-\eta)G_B(t-\eta)(G_B(\eta))^2 G_A(t-\eta) + G_B(t)G_A(t)\int_0^t d\eta\, f_B(t-\eta)G_B(\eta)G_A(t-\eta). \tag{2}$$

Obviously by an interchange of subscripts we may write

$$p_{21}(t) = G_B(t)\int_0^t d\eta\, f_A(t-\eta)G_A(t-\eta)(G_A(\eta))^2 G_B(t-\eta) + G_B(t)G_A(t)\int_0^t d\eta\, f_A(t-\eta)G_A(\eta)G_B(t-\eta). \tag{3}$$

The remainders of all state probability derivations appear in Appendix 1.

Combat Figures of Merit

The transient state probabilities $p_{22}(t)$, $p_{12}(t)$, $p_{21}(t)$, and $p_{11}(t)$ along with the absorbing state probabilities $p_{20}(t)$, $p_{02}(t)$, $p_{10}(t)$, and $p_{01}(t)$ provide all the

information necessary to compute the following commonly used figures of merit.

1. The expected value and standard deviation of the survivors on both side A and side B as a function of time. For side A

$$\begin{aligned} m_A(t) &= 1P[A(t)=1]+2P[A(t)=2] \\ &= 1(p_{10}(t)+p_{11}(t)+p_{12}(t))+2(p_{20}(t)+p_{21}(t)+p_{22}(t)). \end{aligned} \tag{4}$$

$$\text{Computing } E[A^2(t)] = 1P[A(t)=1]+4P[A(t)=2]$$

will then provide

$$\sigma_A(t) = (E[A^2(t)] - m_A^2(t))^{1/2}. \tag{5}$$

Similarly,

$$m_B(t) = 1(p_{01}(t)+p_{11}(t)+p_{21}(t))+2(p_{02}(t)+p_{12}(t)+p_{22}(t) \tag{6}$$

and

$$\sigma_B(t) = (E[B^2(t)] - m_B^2(t))^{1/2}, \tag{7}$$

where

$$E[B^2(t)] = 1P[B(t)=1]+4P[B(t)=2].$$

In particular we get the expected values and standard deviations of the number of survivors on the A side and B side by letting $t \to \infty$ in Eqs. (4)–(7).

2. The expected value and standard deviation of T_D the time duration of combat. These are computed using the well-known integral formulas, derived by an integration by parts, for the first and second moments of a generic non-negative random variable X with finite second moment, and complementary distribution function $G(x)$, namely

$$E[X] = \int_0^\infty G(x)\,dx, \qquad E[X^2] = 2\int_0^\infty xG(x)\,dx.$$

The standard deviation is computed as $(E[X^2]-E^2[X])^{1/2}$. In the case of time duration of combat we use the obvious result that

$$G_{T_D}(t) = p_{22}(t)+p_{21}(t)+p_{12}(t)+p_{11}(t).$$

Thus

$$\mu_{T_D} = \int_0^\infty (p_{22}(t)+p_{21}(t)+p_{21}(t)+p_{12}(t)+p_{11}(t))\,dt \tag{8}$$

and

$$\sigma_{T_D} = (E[T_D^2]-\mu_{T_D}^2)^{1/2} \tag{9}$$

where

$$E[T_D^2] = 2\int_0^\infty t(p_{22}(t) + p_{21}(t) + p_{12}(t) + p_{11}(t))\,dt. \tag{10}$$

3. The probabilities of win by the A side and B side. These two probabilities, $P[A]$ and $P[B]$, respectively, are obviously given by

$$P[A] = \lim_{t\to\infty} (p_{20}(t) + p_{10}(t)) \tag{11}$$

and

$$P[B] = 1 - P[A] = \lim_{t\to\infty} (p_{02}(t) + p_{01}(t)). \tag{12}$$

4. COMPARISONS BETWEEN SOME LANCHESTER MODELS

In this section we present the results of a study making use of the solutions developed in this article. The main purpose of this study was to evaluate how well the classical square law deterministic and exponential interfiring time Lanchester models approximate Models 1.1 and 1.2 (i.e., zero-breakpoint-for-both-sides cases) in situations where we allow either one or both of the sides to have a gamma(2) interfiring time and single-shot kill probability which do not vary from round to round.

The comparisons were motivated by the fact that it is common in combat models where it is known that the interfiring times are not exponential, to assume they are and use the means of the true distributions. Thus, if μ is the mean of the true interfiring time distribution, the killing rate r, is taken to be p/μ (done appropriately for each side) and either the exponential Lanchester (both sides exponentially distributed) or the deterministic Lanchester differential equations are used with the appropriate p/μs as the attrition coefficients. For a further discussion of these matters, see references [2], [6], and [8].

It should be noted that in the literature the exponential interfiring time model is usually referred to as the stochastic Lanchester model. In our view this is bad jargon because any random interfiring time is stochastic in nature. We believe it more accurate to modify the word Lanchester with, for example, constant, exponential, gamma(2), or lognormal in the cases when the interfiring times are constant, exponential, gamma(2), or lognormal random variables, respectively. Using the word Lanchester, however, is appropriate since the basic assumptions of independent firers, random selection of targets, etc., is a common thread. We also reserve the words "deterministic Lanchester" to mean the classical differential equations of combat in which there is no randomness.

A perusal of the general results of the previous section and Appendix 1 shows that the only functions required to compute the various state probabilities are the interkilling time density and complementary distribution functions for sides A and B. The generic versions of these two functions for

the constant round-to-round exponential interfiring time density function and single-shot kill probability are well known and given by

$$f(t) = (p/\mu)e^{-(p/\mu)t}, \quad t \geq 0$$
$$= 0, \quad \text{otherwise,}$$

and

$$G(t) = e^{-(p/\mu)t}, \quad t \geq 0$$
$$= 0, \quad \text{otherwise,}$$

where

$$\mu = \text{mean interfiring time,}$$
$$p = \text{kill probability.}$$

For the gamma(2) interfiring time random variable density function, given by

$$g(t) = (4/\mu^2)te^{-(2/\mu)t}, \quad t \geq 0$$
$$= 0, \quad \text{otherwise,}$$

we compute the interkilling time random variable density function by first writing its distribution function

$$F(t) = P[T \leq t] = \sum_{n=1}^{\infty} P[T \leq t \mid N = n]P[N = n]$$
$$= \sum_{n=1}^{\infty} H_n(t)pq^{n-1},$$

where T = interkilling random variable, N = round number on which the kill occurred, $H_n(t)$ = n-fold convolution of the interfiring time distribution $H(t)$ with itself, p = kill probability, and $q = 1 - p$.

Differentiating gives the density function

$$f(t) = \sum_{n=1}^{\infty} h_n(t)pq^{n-1}, \quad t \geq 0$$
$$= 0, \quad \text{otherwise.}$$

Now the n-fold convolution of a gamma(2) is a gamma($2n$) and is thus given by

$$h_n(t) = \frac{[(2/\mu)]^{2n-1}(2/\mu)e^{-(2/\mu)t}}{(2n-1)!}, \quad t \geq 0$$
$$= 0, \quad \text{otherwise.}$$

so that the above infinite series may be written as

$$f(t) = \frac{2(p/\mu)e^{-(2/\mu)t}}{q^{1/2}} \sum_{n=1}^{\infty} \frac{[(2/\mu)tq^{1/2}]^{2n-1}}{(2n-1)!}, \quad t \geq 0$$
$$= 0, \quad \text{otherwise.}$$

The series on the right is recognized as the hyperbolic sine so that

$$f(t) = \frac{2(p/\mu)e^{-(2/\mu)t}}{q^{1/2}} \sinh[(2/\mu)tq^{1/2}], \quad t \geq 0$$
$$= 0, \quad \text{otherwise.}$$

$f(t)$ may also be written in its exponential form

$$f(t) = \frac{(p/\mu)e^{-(2/\mu)t}}{q^{1/2}} (e^{(2/\mu)tq^{1/2}} - e^{-(2/\mu)tq^{1/2}}), \quad t \geq 0$$
$$= 0, \quad \text{otherwise,}$$

from which we then get by integration that

$$G(t) = \frac{pe^{-(2/\mu)t}}{2q^{1/2}} \left[\frac{e^{(2/\mu)q^{1/2}t}}{1 - q^{1/2}} - \frac{e^{-(2/\mu)q^{1/2}t}}{1 + q^{1/2}} \right], \quad t \geq 0$$
$$= 0, \quad \text{otherwise.}$$

A very important characteristic to note about the two interkilling time random variables is that in the exponential interfiring case the interkilling time depends only on the kill rate $r = p/\mu$ but in the gamma(2) case both the kill rate $p/\mu = r$ and the kill probability p (or mean interfiring time μ) must be specified.

Our study consisted of computing the relative difference that obtains when a figure of merit for a combat is computed using the hypothesized interkilling time random variables for each of the sides and using either the exponential or deterministic approximation. Relative difference (in percent) here is defined as

$$\frac{\theta_G - \theta_A}{\theta_G} \times 100,$$

where θ_G = generic figure of merit in the hypothesized case, and θ_A = the same figure of merit in either the exponential or deterministic approximation to the hypothesized case. Thus, for example, if both sides A and B have gamma(2) interfiring times with parameters $\mu_A = 1/10$, $p_A = 1/10$ (therefore $r_A = 1$) and $\mu_B = 2/5$, $p_B = 1/2$ (therefore $r_B = 5/4$), respectively, then a comparison is made with sides A and B both exponential (or deterministic) with kill rates of $r_A = 1$ and $r_B = 5/4$, respectively. We also considered "mixed" battles such as side A gamma(2), side B exponential and compared it with both the exponen-

tial–exponential and deterministic–deterministic with the appropriate kill rates for each side.

We now discuss briefly the concept of parity and nonparity that we use in our subsequent descriptions. In the deterministic Lanchester square law model, parity occurs whenever $r_A(a_0^2 - a_f^2) = r_B(b_0^2 - b_f^2)$; the battle goes to infinity and neither side wins. *We continue to use the same definition of parity in all stochastic cases using* $r = p/\mu$. *Strict parity* is defined as all parameters being identical on both sides, i.e., the initial numbers, kill probabilities, and interfiring time random variables. And when that happens $P[A] = P[B] = 1/2$. It should be noted that in the exponential–exponential battles when $a_0 = b_0$ and $a_f = b_f$ then parity is equivalent to strict parity.

The computations for the state probabilities were made at White Sands Missile Range on the Sperry 1100/82 system. A Gaussian quadrature technique was used to evaluate all the integrals involved, not only for the state probabilities but the various figures of merit. The particular quadrature technique we used is described in reference [10].

The accuracy of an integration over a time interval $[0, t]$ is a function of the number of equal length segments the interval is decomposed into and the degree of the polynomial used for each segment. We used three segments and a 19th-degree polynomial. State probabilities were computed at 30 times points in the interval $[0, t_\infty]$, where t_∞ is essentially $t = \infty$. Thus, if one were to compute the state probabilities at t_∞, the sum of the transient probabilities

$$p_{22}(t_\infty) + p_{21}(t_\infty) + p_{12}(t_\infty) + p_{11}(t_\infty) \doteq 0$$

and the sum of the absorbing probabilities

$$p_{02}(t_\infty) + p_{01}(t_\infty) + p_{10}(t_\infty) + p_{20}(t_\infty) \doteq 1.$$

CPU times were the largest whenever gamma(2)–gamma(2) combats were run, i.e., the A side and B side each had gamma(2) interfiring times. And, for these cases, the CPU times required to compute all $p_{ij}(t)$s, $m_A(t)$, $\sigma_A(t)$, $m_B(t)$, $\sigma_B(t)$ and the eight overall figures of merit μ_{T_D}, σ_{T_D}, $E[A(\infty)]$, $\sigma[A(\infty)]$, $E[B(\infty)]$, $\sigma[B(\infty)]$, $P[A]$, and $P[B]$ were approximately 3 and 2 hours for "reselect on" and "reselect off," respectively.

Typical time varying characteristics are shown in Figures 6 and 7. In this particular example we have "reselect on" (Model 1.1) and gamma(2) interfiring for both sides A and B with firing rates $1/\mu_A = 10$ and $1/\mu_B = 10/9$. Setting $p_A = 1/10$ and $p_B = 9/10$ gives $r_A = 1$ and $r_B = 1$. It should be noted that in this case we took $t_\infty = 3.5$; thus the sum of the transient state probabilities at $t_\infty = 3.5$, in Figure 6(a), is approximately 0 and the sum of the absorbing state probabilities at $t_\infty = 3.5$, in Figure 6(b), is approximately 1. The 30 time points at which these state probabilities were computed, as shown in Figure 6, were selected so that a Gaussian quadrature could be used for computing $E[T_D]$ and $\sigma[T_D]$ from Eqs. (8) and (10), wherein the upper limit of ∞ is replaced by $t_\infty = 3.5$.

It should be noted here that the results presented in Figures 6 and 7 are for a situation in which parity, but not strict parity, obtains; and for the hypo-

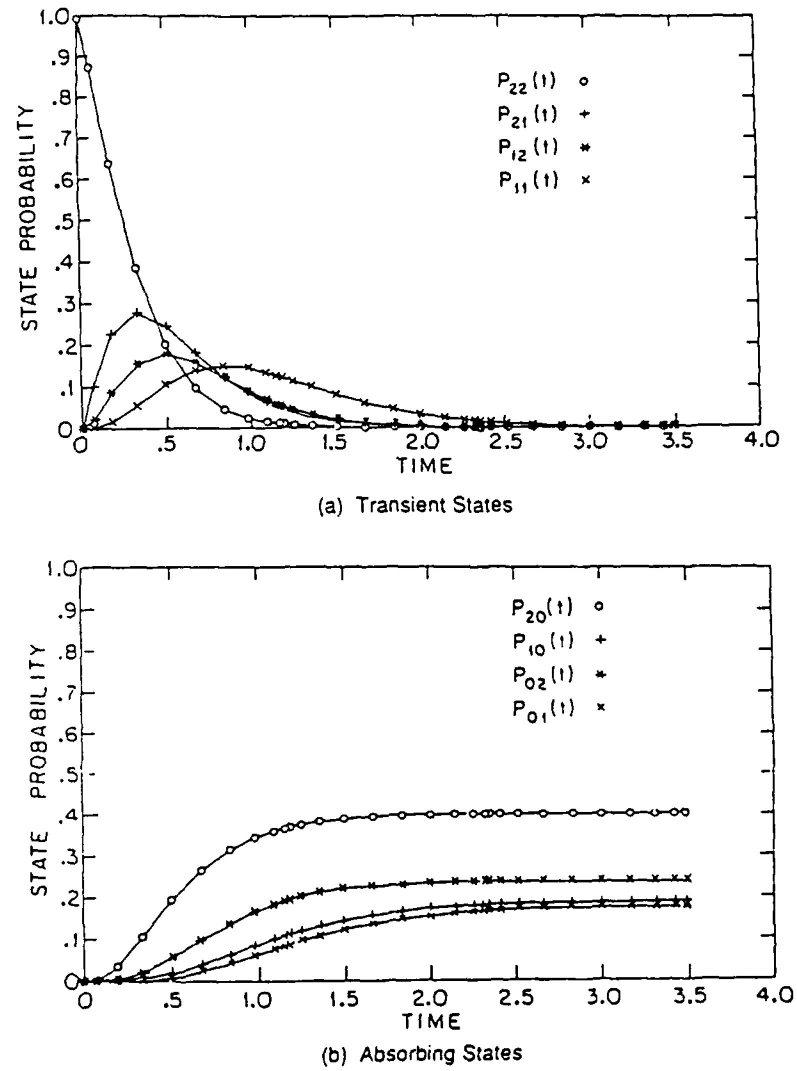

Figure 6. State probabilities versus time for a Model 1.1 combat; A side gamma(2) with $1/\mu_A = 10$, $p_A = 1/10$, $r_A = 1$; B side gamma(2) with $1/\mu_B = 10/9$, $p_B = 9/10$, $r_B = 1$; reselect on.

thesized interkilling time random variables

$$P[A] = p_{20}(\infty) + p_{10}(\infty) \doteq p_{20}(t_\infty) + p_{10}(t_\infty) = 0.5892,$$

$$P[B] = p_{02}(\infty) + p_{01}(\infty) \doteq p_{02}(t_\infty) + p_{10}(t_\infty) = 0.4106.$$

However, for the equivalent exponential–exponential combat, where $r_A = r_B = 1$, the situation is one of strict parity and $P[A] = P[B] = 1/2$. Figures 8 and 9 show the exponential–exponential result corresponding to Figures 6 and 7,

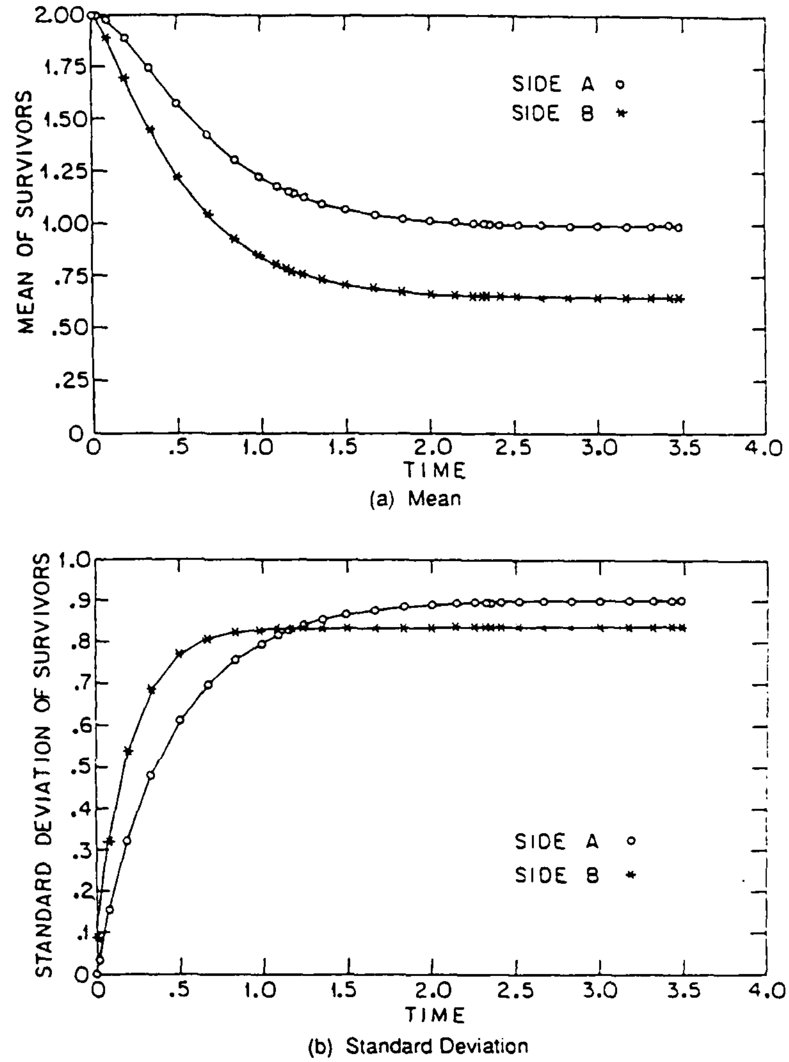

Figure 7. Mean and standard deviation of number of survivors versus time for a Model 1.1 combat; A side gamma(2) with $1/\mu_A = 10$, $p_A = 1/10$, $r_A = 1$; B side gamma(2) with $1/\mu_B = 10/9$, $p_B = 9/10$, $r_B = 1$; reselect on.

respectively. Note the significant differences that occur in the time-varying characteristics between the hypothesized stochastic and exponential models.

The differences between the hypothesized model and deterministic Lanchester models are even greater than those obtained when the hypothesized model is compared to the exponential model. In fact, for the case just discussed, in the equivalent deterministic case the battle completion time is infinite and neither side wins (since both sides go to annihilation). In Table 2 we present, for this case, the values of the eight overall battle figures of merit for each of the three models and the relative differences that are obtained

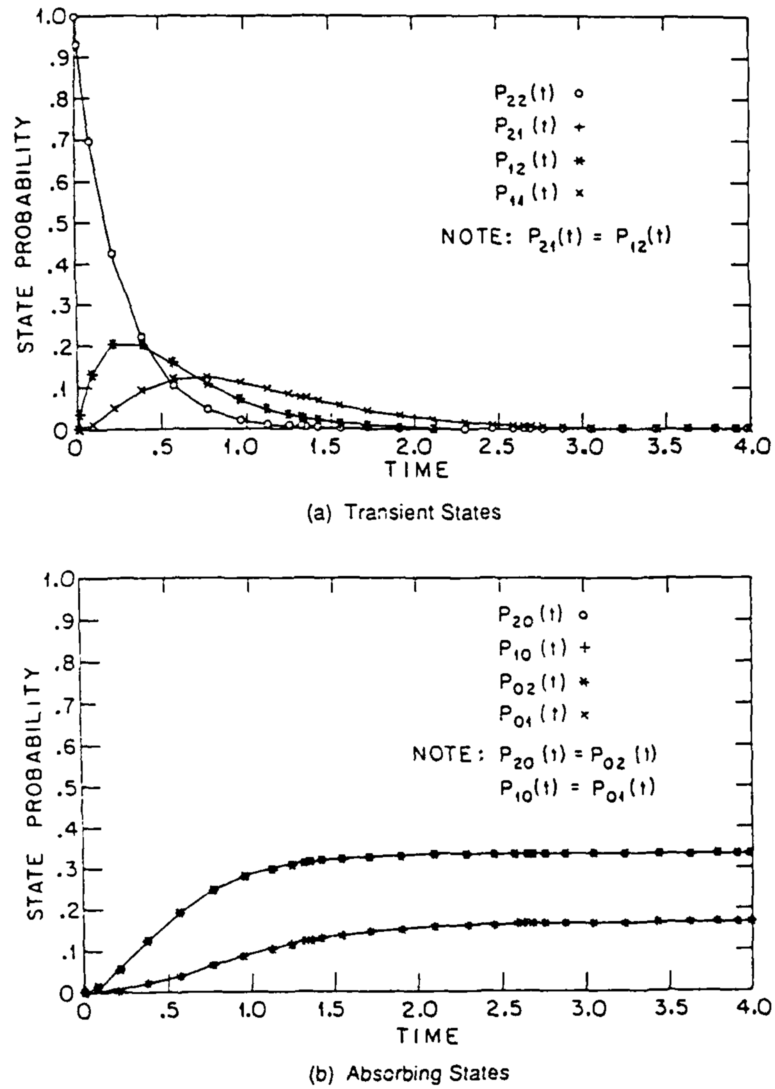

Figure 8. State probabilities versus time for an exponential 2-on-2; A side with $r_A = 1$; B side with $r_B = 1$.

when compared to the hypothesized model. Obviously the differences shown are large and point to the importance of developing these models in order to obtain greater realism.

It should be noted here that in our simulation studies, referred to in Section 1, we are running much larger combats, for example, 100-on-50, with various interfiring time random variables including the gamma(2). We have found there that as the battle size grows, i.e., as we increase the initial numbers involved, the differences between the hypothesized model and either the exponential or deterministic approximations gets larger percentagewise also.

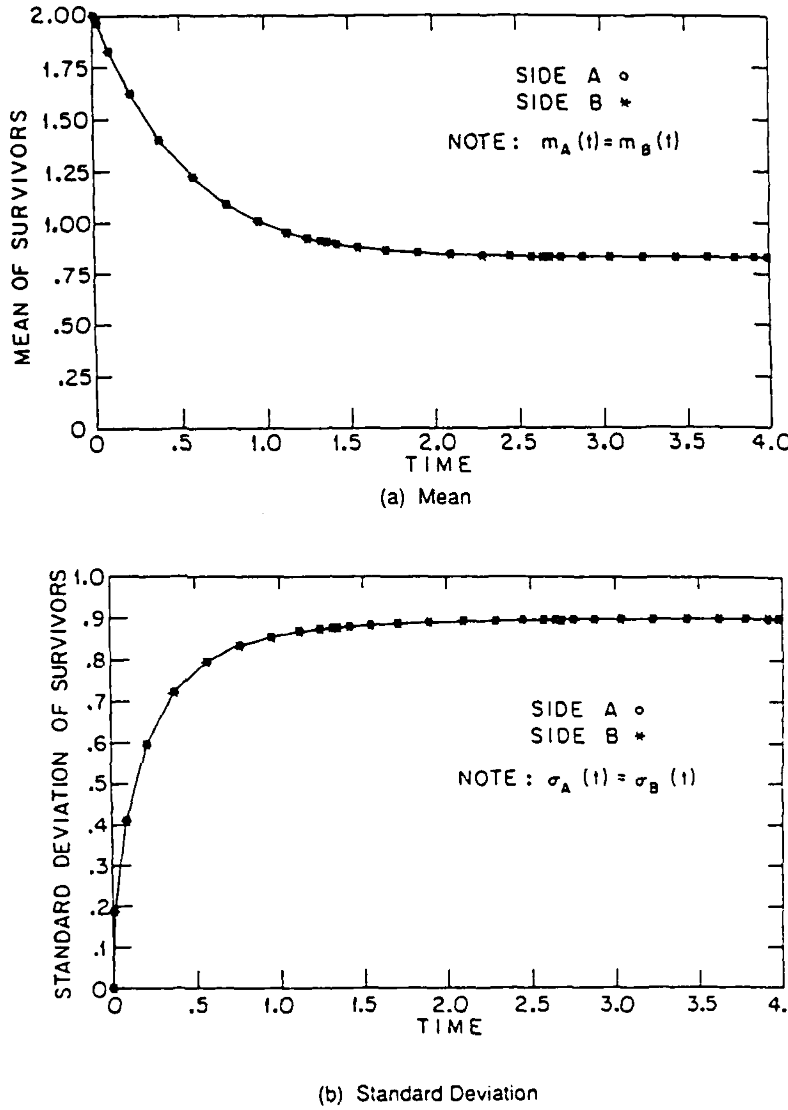

Figure 9. Mean and standard deviation of number of survivors versus time for an exponential 2 of 2; A side with $r_A = 1$; B side with $r_B = 1$.

What is being emphasized here is that we are comparing the hypothesized with either the exponential or the deterministic approximation and *not* differences between the exponential and deterministic models (although large differences there also exist; see reference [2]).

In Appendix 2 is a brief description of all the cases run and tables of relative differences that were obtained. A perusal of those tables will show relative differences, when compared to the exponential model, as high as 44 percent.

Table 2. Overall figures of merit and relative differences (in percent) when the hypothesized model is compared with the exponential and deterministic Lanchester models. For the hypothesized model; side A gamma(2), $1/\mu_A = 10$, $p_A = 1/10$, $r_A = 1$; side B gamma(2), $1/\mu_A = 10/9$, $p_B = 9/10$, $r_B = 1$.

Figure of merit	Hypothesized model	Exponential	Deterministic	Relative difference (%) (exponential)	Relative difference (%) (deterministic)
$E[T_D]$	0.8665	0.7497	∞	+13.480	$-\infty$
$\sigma[T_D]$	0.5359	0.5568	0	−3.900	100
$E[A(\infty)]$	0.9919	0.8337	0	+15.949	100
$\sigma[A(\infty)]$	0.9016	0.8974	0	+0.466	100
$E[B(\infty)]$	0.6476	0.8337	0	−28.737	100
$\sigma(B(\infty)]$	0.8376	0.8974	0	−7.139	100
$P[A]$	0.5892	0.4997	0	+15.190	100
$P[B]$	0.4106	0.4997	0	−21.700	100

6. CONCLUSIONS

Two versions of some stochastic homogeneous two-on-two combat models have been defined and state probabilities have been derived for each. In the first version, a marksman whose target is killed resumes afresh the killing process on a surviving target; in the second version, the marksman whose target is killed merely uses up his remaining time to kill on a surviving target. The state probabilities in turn were used to develop four time-varying characteristics $m_A(t)$, $\sigma_A(t)$, $m_B(t)$, $\sigma_B(t)$ and eight overall battle characteristics μ_{T_D}, σ_{T_D}, $E[A(\infty)]$, $\sigma[A(\infty)]$, $E[B(\infty)]$, $\sigma[B(\infty)]$, $P[A]$, and $P[B]$.

Comparisons were made, in terms of relative difference, with equivalent exponential and deterministic Lanchester models. It was found that both the exponential and deterministic Lanchester models are very poor approximations of the hypothesized model. In fact, among the cases we considered, there was a relative difference of 44 percent in the figure of merit $E[A(\infty)]$ when compared to the exponential model. And in this study we considered only one interfiring time that was not exponential; namely, the gamma(2). One can reasonably conjecture that larger relative differences would surface when other distributions are considered. But the point is that we have demonstrated that the exponential approximation is indeed a poor one and that further work must be done to develop the theory of small-to-moderate-size stochastic combat models.

APPENDIX 1

Model 1.1 (Continued)

3. $p_{11}(t)$. We now consider $p_{11}(t)$ and write it as

$$p_{11}(t) = p_{11}^{(1)}(t) + p_{11}^{(2)}(t) + p_{11}^{(3)}(t) + p_{11}^{(4)}(t). \tag{13}$$

Clearly, once we write the first two functions on the right-hand side, we may

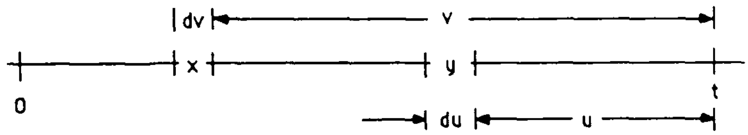

Figure 10. Definition of the variables u and v for the computation of $p_{11}^{(1)}(t)$ and $p_1^{(2)}(t)$. x = a B kills an A in $(t-v-dv, t-v)$; y = the surviving A kills a B in $(t-v-du, t-u)$; $u < v$.

set down immediately the second two by symmetry. Consider now Figure 10 above which shows that a B kills an A first on the interval $(t-v-dv, t-v)$, the surviving A then kills a B in $(t-u-du, t-u)$ with no subsequent killing until after time t.

Now $p_{11}^{(1)}(t)$ will be written as

$$p_{11}^{(1)}(t) = \int_0^t dv \int_0^v du\, p_{11}^{(1)}(t, u, v),$$

where $p_{11}^{(1)}(t, u, v)\, du\, dv$ = probability that both Bs are aiming at the same A, one of the Bs kills an A in $(t-v-dv, t-v)$, the surviving A kills a B in $(t-u-du, t-u)$, and there is no other killing until after time t. $p_{11}^{(1)}(t, u, v)\, du\, dv$ may now be written as 1/2 times the product of

$2f_B(t-v)\,dv$ = probability that one of the two Bs kills an A in the time interval $(t-v-dv, t-v)$,
$G_B(t-v)$ = probability that nonkilling B has a kill time $> t-v$,
$G_B(v-u)$ = probability that the B which will be killed by the surviving A reaims with an interkill time $> v-u$,
$G_B(v)$ = probability that the B which will survive reaims with an interkill time $> v$,
$G_A(t-v)$ = probability that killed A had a time to kill $> t-v$,
$f_A(t-u)\,du$ = probability that the surviving A kills a B in $(t-u-du, t-u)$,

and finally

$G_A(u)$ = the probability that the killing A reaims with an interkill time $> u$.

So we have

$$p_{11}^{(1)}(t) = \int_0^t dv \Big[f_B(t-v) G_B(t-v) G_B(v) G_A(t-v) \times \int_0^v du\, f_A(t-u) G_A(u) G_B(v-u) \Big]. \tag{14}$$

Now we consider $p_{11}^{(2)}(t)$ and write it as

$$p_{11}^{(2)}(t) = \int_0^t dv \int_0^v du\, p_{11}^{(2)}(t, u, v).$$

Given the set of all aiming configurations in which each B aims at a different A, it is easy to see that within this set half the time the surviving A kills the killer B and half the time the surviving A kills the nonkiller B. So we break up $p_{11}^{(2)}(t, u, v)$ into two terms. The first term arises when the surviving A kills the killer and the second term arises when the surviving A kills the nonkiller. Thus the final result becomes

$$\begin{aligned} p_{11}^{(2)}(t, u, v)\, du\, dv = (1/2)^2[&2 f_B(t-v)\, dv\, G_B(v-u) G_B(t) G_A(t-v) \\ &\times f_A(t-u)\, du\, G_A(u) + 2 f_B(t-v)\, dv\, G_B(v) G_B(t-u) \\ &\times G_A(t-v) f_A(t-u)\, du\, G_A(u)], \end{aligned}$$

in which the factors common to both terms in the brackets are

$2f_B(t-v)\,dv$ = probability one of the two Bs kills an A in the time interval $(t-v-dv, t-v)$,
$G_A(t-v)$ = probability that killed A had a time to kill $> t-v$,
$f_A(t-u)\,du$ = probability that the surviving A kills a B in $(t-u-du, t-u)$,
$G_A(u)$ = probability that the killing A reaims with an interkill time $> u$,

whereas the second and third factors in each term are unique to the situation. In the first term

$G_B(v-u)$ = probability killer B reaims and has interkill time greater than $v-u$ [since he gets killed in the interval $(t-u-du, t-u)$],
$G_B(t)$ = probability nonkiller B (or the surviving B) has an interkill time $> t$,

and in the second term

$G_B(v)$ = probability killer B (or the surviving B) reaims and has interkill time $> v$,
$G_B(t-u)$ = probability the nonkiller B has an interkill time $> t-u$ [since he gets killed in the interval $(t-u-du, t-u)$].

So finally

$$\begin{aligned} p_{11}^{(2)}(t) = \frac{1}{2}\Big[G_B(t) \int_0^t dv \Big[f_B(t-v) G_A(t-v) \int_0^v du\, f_A(t-u) G_A(u) G_B(v-u) \Big] \\ + \int_0^t dv \Big[f_B(t-v) G_B(v) G_A(t-v) \int_0^v du\, G_B(t-u) f_A(t-u) G_A(u) \Big]\Big]. \end{aligned} \tag{15}$$

To complete the computation for $p_{11}(t)$ we must write expressions for $p_{11}^{(3)}(t, u, v)$ and $p_{11}^{(4)}(t, u, v)$. These two arise from a transition from (2, 1) to (1, 1) in which initially either both As are aiming at the same B or they are not. In Figure 11 below we show that an A kills a B first in the interval $(t-u-du, t-u)$, the surviving B then kills an A in $(t-v-dv, t-v)$ with no subsequent killing until after time t. Clearly we may immediately write down $p_{11}^{(3)}(t)$ and $p_{11}^{(4)}(t)$ by interchanging in equations (14) and (15), respectively, A

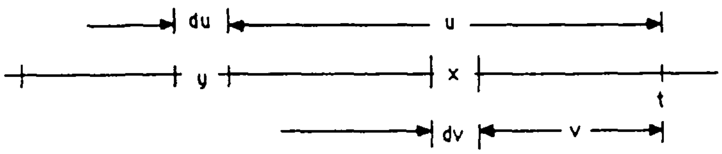

Figure 11. Definition of the variables u and v for the computations of $p_{11}^{(3)}(t)$ and $p_{11}^{(4)}(t)$. y = an A kills a B in $(t-u-du,\ t-u)$; x = the surviving B kills an A in $(t-v-dv,\ v-u)$; $u>v$.

with B and u with v. Thus

$$p_{11}^{(3)}(t)=\int_0^t du\left[f_A(t-u)G_A(t-u)G_A(u)G_B(t-u)\right.$$
$$\left.\times\int_0^u dv\, f_B(t-v)G_B(v)G_A(u-v)\right], \tag{16}$$

and

$$p_{11}^{(4)}(t)=\frac{1}{2}\left[G_A(t)\int_0^t du\left[f_A(t-u)G_B(t-u)\int_0^u dv\, f_B(t-v)G_B(v)G_A(u-v)\right]\right.$$
$$\left.+\int_0^t du\left[f_A(t-u)G_A(u)G_B(t-u)\int_0^u dv\, G_A(t-v)f_B(t-v)G_B(v)\right]\right]. \tag{17}$$

It should be noted that in the expressions $p_{11}^{(i)}(t, u, v)$, $i=1,2,3,4$ it is not always the case that u and v are the backward recurrence times, measured at time t, of the surviving A and B, respectively. Consider $p_{11}^{(2)}(t, u, v)$ which arises from an initial configuration with each B aiming at a different A. At time t, one of two situations obtain. Either the surviving B killed an A or it did not. If it did kill an A, its backward recurrence time is v; if it did not kill an A its backward recurrence time is t.

Summarizing to this point, Eqs. (1)–(3) and (13)–(17) give all the transient state probabilities, i.e., each of states has in the limit as $t\to\infty$ zero probability; and we now turn our attention to writing the probabilities associated with the four absorbing states (0, 2), (2, 0), (0, 1), and (1, 0).

4. $p_{02}(t)$, $p_{20}(t)$. We now consider $p_{02}(t)$ and write it as

$$p_{02}(t)=p_{02}^{(1)}(t)+p_{02}^{(2)}(t). \tag{18}$$

Consider Figure 12 below which shows that a B kills an A first in the interval $(t-\zeta-d\zeta,\ t-\zeta)$ followed by another kill of the surviving A by a B in the interval $(t-\eta-d\eta,\ t-\eta)$. Now $p_{02}^{(1)}(t)$ will be written as

$$p_{02}^{(1)}(t)=\int_0^t d\zeta\int_0^\zeta d\eta\, p_{02}^{(1)}(t, \zeta, \eta),$$

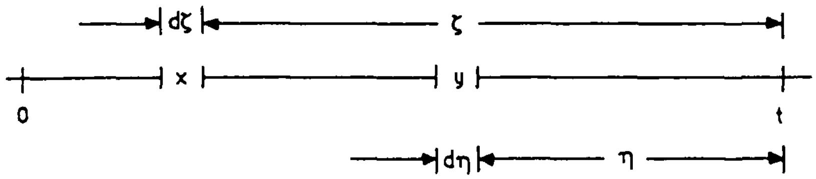

Figure 12. Definition of the variables ζ and η for the computations of $p_{02}^{(1)}(t)$ and $p_{02}^{(2)}(t)$. $x =$ a B kills an A in $(t-\zeta-d\zeta,\ t-\zeta)$; $y =$ a B kills another A in $(t-\eta-d\eta,\ t-\eta)$; $\zeta > \eta$.

where $p_{02}^{(1)}(t, \zeta, \eta)\, d\zeta\, d\eta =$ probability that both Bs are aiming at the same A, one of the Bs kills an A in $(t-\zeta-d\zeta, t-\zeta)$ and then one of the Bs kills the surviving A in $(t-\eta-d\eta, t-\eta)$. $p_{02}^{(1)}(t, \zeta, \eta)\, d\zeta\, d\eta$ may now be written as 1/2 times the product of

$2f_B(t-\zeta)\, d\zeta =$ probability that one of the two Bs kills an A in the time interval $(t-\zeta-d\zeta, t-\zeta)$,

$G_B(t-\zeta) =$ probability that nonkilling B has a kill time $> t-\zeta$,

$2f_B(\zeta-\eta)\, d\eta =$ probability, after both Bs reaim, that a B kills the surviving A in the time interval $(\zeta-\eta-d\eta, \zeta-\eta)$,

$G_B(\zeta-\eta) =$ probability that nonkilling B has an interkill time $> \xi-\eta$,

$G_A(t-\zeta) =$ probability that the first A that was killed had a kill time $> t-\zeta$,

and finally $G_A(t-\eta) =$ probability that the second A that was killed had a kill time $> t-\eta$. So we have

$$p_{02}^{(1)}(t) = 2\int_0^t d\zeta \Big[f_B(t-\zeta) G_B(t-\zeta) G_A(t-\zeta) \times \int_0^\zeta d\eta\, f_B(\zeta-\eta) G_B(\zeta-\eta) G_A(t-\eta) \Big]. \tag{19}$$

Now consider $p_{02}^{(2)}(t)$ and write it as

$$p_{02}^{(2)}(t) = \int_0^t d\zeta \int_0^\zeta d\eta\, p_{02}^{(2)}(t, \zeta, \eta).$$

Here we must break up $p_{02}^{(2)}(t, \zeta, \eta)$ into two terms. The first term arises when the same B kills both As and the second term arises when each B kills an A. The final result becomes

$$p_{02}^{(2)}(t, \zeta, \eta)\, d\zeta\, d\eta = 1/2[2f_B(t-\zeta)\, d\zeta\, f_B(\zeta-\eta)\, d\eta\, G_B(t-\eta) G_A(t-\zeta) G_A(t-\eta) + 2f_B(t-\zeta)\, d\zeta\, f_B(t-\eta)\, d\eta\, G_B(\zeta-\eta) G_A(t-\zeta) G_A(t-\eta)]$$

in which the factors common to both terms in the brackets are,

$2f_B(t-\zeta)\, d\zeta =$ probability one of the two Bs kills the first A in the time interval $(t-\zeta-d\zeta, t-\zeta)$,

$G_A(t-\zeta)$ = probability that the first killed A had a time to kill $> t-\zeta$,
$G_A(t-\eta)$ = probability that the second killed A had a time to kill $> t-\eta$,

whereas the factors not common are unique to the situation. In the first term

$f_B(\zeta-\eta)\,d\eta$ = probability that the killing B reaims and kills the surviving A in the time interval $(t-\eta-d\eta,\ t-\eta)$,
$G_B(t-\eta)$ = probability that the B who kills no As has a kill time $> t-\eta$,

and in the second term

$f_B(t-\eta)\,d\eta$ = probability that the other B (who did not have to reaim) kills the surviving A in the interval $(t-\eta-d\eta,\ t-\eta)$,
$G_B(\zeta-\eta)$ = probability that the first B who killed an A (and had to reaim) has an interkill time $> \zeta-\eta$.

So, finally,

$$p_{02}^{(2)}(t) = \int_0^t d\zeta \left[f_B(t-\zeta)G_A(t-\zeta) \int_0^\zeta d\eta\, G_B(t-\eta) f_B(\zeta-\eta) G_A(t-\eta) \right] + \int_0^t d\zeta \left[f_B(t-\zeta)G_A(t-\zeta) \int_0^\zeta d\eta\, G_B(\zeta-\eta) f_B(t-\eta) G_A(t-\eta) \right]. \tag{20}$$

Similarly, write

$$p_{20}(t) = p_{20}^{(3)}(t) + p_{20}^{(4)}(t). \tag{21}$$

If we suppose that the first B is killed in $(t-\zeta-d\zeta,\ t-\zeta)$ and the second B is killed in $(t-\eta-d\eta,\ t-\eta)$, then by simply interchanging A with B in Eqs. (19) and (20) we may get $p_{20}^{(3)}(t)$ and $p_{20}^{(4)}(t)$. Thus

$$p_{20}^{(3)}(t) = 2\int_0^t d\zeta \left[f_A(t-\zeta)G_A(t-\zeta)G_B(t-\zeta) \times \int_0^\zeta d\eta\, f_A(\zeta-\eta)G_A(\zeta-\eta)G_B(t-\eta) \right] \tag{22}$$

and

$$p_{20}^{(4)}(t) = \int_0^t d\zeta \left[f_A(t-\zeta)G_B(t-\zeta) \int_0^\zeta d\eta\, G_A(t-\eta) f_A(\zeta-\eta) G_B(t-\eta) \right] + \int_0^t d\zeta \left[f_A(t-\zeta)G_B(t-\zeta) \int_0^\zeta d\eta\, G_A(\zeta-\eta) f_A(t-\eta) G_B(t-\eta) \right]. \tag{23}$$

5. $p_{01}(t)$, $p_{10}(t)$. We next consider $p_{01}(t)$ and write it as

$$p_{01}(t) = p_{01}^{(1)}(t) + p_{01}^{(2)}(t) + p_{01}^{(3)}(t) + p_{01}^{(4)}(t). \tag{24}$$

To develop expressions for the terms on the right-hand side of Eq. (24) it is very convenient to work with the backward recurrence time and its associated instantaneous kill rate. For the first term, recall both Bs are initially aiming at the same A and refer back to Figure 10 and consider the following sequence of events:

1. One of the Bs killed an A in the time interval $(t-v-dv, t-v)$.
2. Then both Bs reaimed and one of the Bs was subsequently killed by the surviving A in the time interval $(t-u-du, t-u)$.

Therefore, at time t the surviving A has a backward recurrence time of u with instantaneous kill rate $r_A(u)$ and the surviving B has a backward recurrence time of v with instantaneous kill rate $r_B(v)$.

Consider now the time interval $(t, t+\Delta)$ and, in the usual fashion, write an expression for $p_{01}^{(1)}(t+\Delta)$ retaining only the first order terms in Δ. Thus the probability of being in state $(0, 1)^{(1)}$ at time $t+\Delta$ is equal to the sum of two probabilities, namely, (1) the probability of being in state $(0, 1)^{(1)}$ at time t times the probability of remaining there in $(t, t+\Delta)$ [which is one since $(0, 1)^{(1)}$ is an absorbing state], and (2) the probability $p_{11}^{(1)}(t, u, v)\, du\, dv$, of being in the state $(1, 1)^{(1)}$ at time t with backward recurrence times of u and v for survivors on the A side and B side, respectively, times the probability, $r_B(v)\Delta(1-r_A(u)\Delta)$, that the B-side survivor kills the A and A fails to kill. Thus, we get taking into account all (u, v) pairs,

$$p_{01}^{(1)}(t+\Delta) = p_{01}^{(1)}(t) + \int_0^t dv \int_0^v du\, p_{11}^{(1)}(t, u, v) r_B(v)\Delta(1 - r_A(u)\Delta).$$

Rearranging terms, dividing by Δ, and letting $\Delta \to 0$, we have that

$$dp_{01}^{(1)}(t)/dt = \int_0^t dv \int_0^v du\, p_{11}^{(1)}(t, u, v) r_B(v).$$

Using the initial condition $p_{01}^{(1)}(0) = 0$, we may write

$$p_{01}^{(1)}(t) = \int_0^t d\zeta \int_0^\zeta dv \int_0^v du\, p_{11}^{(1)}(\zeta, u, v) r_B(v).$$

Finally, after substituting in for $r_B(v) = f_B(v)/G_B(v)$ and $p_{11}^{(1)}(\zeta, u, v)$ [see Eq. (14)], we get

$$p_{01}^{(1)}(t) = \int_0^t d\zeta \int_0^\zeta dv \left[f_B(v) f_B(\zeta - v) G_B(\zeta - v) G_A(\zeta - v) \times \int_0^v du\, G_B(v-u) f_A(\zeta - u) G_A(u) \right]. \quad (25)$$

Proceeding in the above manner, i.e., making use of the backward recurrence times and the instantaneous kill rate associated with each, it is easy to

get the next three equations, namely,

$$p_{01}^{(2)}(t) = \frac{1}{2}\left\{\int_0^t d\zeta \left[f_B(\zeta) \int_0^\zeta dv \left[f_B(\zeta - v) G_A(\zeta - v) \right.\right.\right.$$
$$\left.\left. \times \int_0^v du\, G_B(v-u) f_A(\zeta - u) G_A(u) \right]\right]$$
$$+ \int_0^t d\zeta \int_0^\zeta dv \left[f_B(v) f_B(\zeta - v) G_A(\zeta - v) \right.$$
$$\left.\left. \times \int_0^v du\, G_B(\zeta - u) f_A(\zeta - u) G_A(u) \right]\right\}, \tag{26}$$

$$p_{01}^{(3)}(t) = \int_0^t d\zeta \int_0^\zeta du \left[f_A(\zeta - u) G_A(\zeta - u) G_A(u) G_B(\zeta - u) \right.$$
$$\left. \times \int_0^v dv\, f_B(\zeta - v) f_B(v) G_A(u - v) \right], \tag{27}$$

and

$$p_{01}^{(4)}(t) = \frac{1}{2}\left\{\int_0^t d\zeta \left[G_A(\zeta) \int_0^\zeta du \left[f_A(\zeta - u) G_B(\zeta - u) \right.\right.\right.$$
$$\left.\left. \times \int_0^u dv\, f_B(v) f_B(\zeta - v) G_A(u - v) \right]\right]$$
$$+ \int_0^t d\zeta \int_0^\zeta du \left[f_A(\zeta - u) G_A(u) G_B(\zeta - u) \right.$$
$$\left.\left. \times \int_0^u dv\, G_A(\zeta - v) f_B(\zeta - v) f_B(v) \right]\right\}. \tag{28}$$

We next write the right-hand terms of

$$p_{10}(t) = p_{10}^{(1)}(t) + p_{10}^{(2)}(t) + p_{10}^{(3)}(t) + p_{10}^{(4)}(t). \tag{29}$$

Each of these are easily written by taking the appropriate $p_{01}^{(i)}(t)$, $i = 1, 2, 3, 4$ and interchanging A with B and u with v. Table 3 below lists these identifications. Thus to get $p_{10}^{(3)}(t)$ interchange A and B and u with v in $p_{01}^{(1)}(t)$, etc. These identifications result in the formulas

$$p_{10}^{(1)}(t) = \int_0^t d\zeta \int_0^\zeta dv \left[f_B(\zeta - v) G_B(\zeta - v) G_B(v) G_A(\zeta - v) \right.$$
$$\left. \times \int_0^v du\, f_A(\zeta - u) f_A(u) G_B(v - u) \right], \tag{30}$$

$$p_{10}^{(2)}(t) = \frac{1}{2}\left\{\int_0^t d\zeta \left[G_B(\zeta) \int_0^\zeta dv \left[f_B(\zeta - v) G_A(\zeta - v) \right.\right.\right.$$
$$\left.\left. \times \int_0^v du\, f_A(u) f_A(\zeta - u) G_B(v - u) \right]\right]$$
$$+ \int_0^t d\zeta \int_0^\zeta dv \left[f_B(\zeta - v) G_B(v) G_A(\zeta - v) \right.$$
$$\left.\left. \times \int_0^v du\, G_B(\zeta - u) f_A(\zeta - u) f_A(u) \right]\right\}, \tag{31}$$

$$p_{10}^{(3)}(t) = \int_0^t d\zeta \int_0^\zeta du \left[f_A(u) f_A(\zeta - u) G_A(\zeta - u) G_B(\zeta - u) \right.$$
$$\left. \times \int_0^u dv\, G_A(u - v) f_B(\zeta - v) G_B(v) \right], \tag{32}$$

and

$$p_{10}^{(4)}(t) = \frac{1}{2}\left\{\int_0^t d\zeta \left[f_A(\zeta) \int_0^\zeta du \left[f_A(\zeta - u) G_B(\zeta - u) \right.\right.\right.$$
$$\left.\left. \times \int_0^u dv\, G_A(u - v) f_B(\zeta - v) G_B(v) \right]\right]$$
$$+ \int_0^t d\zeta \int_0^\zeta du \left[f_A(u) f_A(\zeta - u) G_B(\zeta - u) \right.$$
$$\left.\left. \times \int_0^u dv\, G_A(\zeta - v) f_B(\zeta - v) G_B(v) \right]\right\}. \tag{33}$$

Table 3. Identifications used to get $p_{10}(t)$ from $p_{01}(t)$.

$(1,1)\rightarrow(0,1)$		$(1,1)\rightarrow(1,0)$	
$p_{01}^{(i)}(t)$	Initial aiming configuration and winning side	$p_{10}^{(i)}(t)$	Initial aiming configuration and winning side
$p_{01}^{(1)}(t)$	Both *B*s aiming at same *A* and *B* side wins	$p_{10}^{(3)}(t)$	Both *A*s aiming at same *B* and *A* side wins
$p_{01}^{(2)}(t)$	Each *B* aiming at a different *A* and *B* side wins	$p_{10}^{(4)}(t)$	Each *A* aiming at a different *B* and *A* side wins
$p_{01}^{(3)}(t)$	Both *A*s aiming at same *B* and *B* side wins	$p_{10}^{(1)}(t)$	Both *B*s aiming at the same *A* and *A* side wins
$p_{01}^{(4)}(t)$	Each *A* aiming at a different *B* and *B* side wins	$p_{10}^{(2)}(t)$	Each *B* aiming at a different *A* and *A* side wins

Model 1.2 ($a_0 = 2$, $a_f = 0$; $b_0 = 2$, $b_f = 0$; **"Reselect Off"**)

The reader is reminded that in Version 2 of Model 1 (Model 1.2), i.e., "reselect off," if a marksman's target is killed by the other member on the side, his remaining time to a firing (or a killing) is carried over to the survivor (if there is one). To develop the state probabilities for this model it is no longer necessary to break down the analysis into terms of the initial aiming configurations, i.e., both Bs aiming at the same A, etc. However, we still do decompose state (1, 1) into states $(1, 1)^{(1)}$, $(1, 1)^{(2)}$, $(1, 1)^{(3)}$, and $(1, 1)^{(4)}$ and these are defined in Figure 13 below.

1. $p_{22}(t)$. As in Model 1.1, we have

$$p_{22}(t) = (G_A(t))^2(G_B(t))^2. \tag{34}$$

2. $p_{12}(t)$, $p_{21}(t)$. Referring back to Figure 5 we define $p_{12}(t, \eta)\, d\eta$ = one of the Bs kills an A in $(t - \eta - d\eta, t - \eta)$, and there are no killings until beyond t. Now

$$p_{12}(t, \eta)\, d\eta = 2 f_B(t - \eta)\, d\eta\, G_B(\eta) G_A(t - \eta) G_A(t),$$

where

$2 f_B(t - \eta)\, d\eta$ = probability that one of the two Bs kills an A in the time interval $(t - \eta - d\eta, t - \eta)$.
$G_B(\eta)$ = probability that the B that killed reaimed and has an interkill time $> \eta$,
$G_B(t)$ = probability that the nonkiller B, since he just carries over his remaining time to kill if he is aiming at the killed A, has a time to kill $> t$,
$G_A(t - \eta)$ = probability that killed A had a time to kill $> t - \eta$,

and finally $G_A(t)$ = probability that surviving A had a time to kill $> t$. Thus we get

$$p_{12}(t) = \int_0^t d\eta\, p_{12}(t, \eta) = 2 G_B(t) G_A(t) \int_0^t d\eta\, f_B(t - \eta) G_B(\eta) G_A(t - \eta). \tag{35}$$

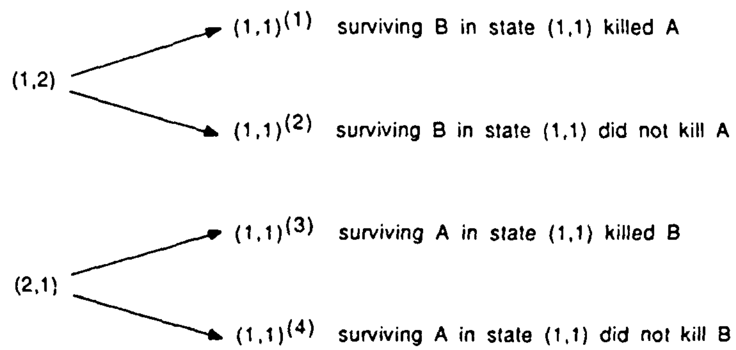

Figure 13. Definition of the states $1, 1)^{(1)}$, $(1, 1)^{(2)}$, $(1, 1)^{(3)}$, and $(1, 1)^{(4)}$ in the "reselect off" model.

Again, by an interchange of A with B, we get

$$p_{21}(t) = 2G_A(t)G_B(t)\int_0^t d\zeta f_A(t-\zeta)G_A(\zeta)G_B(t-\zeta). \tag{36}$$

3. $p_{11}(t)$. As described above we shall write $p_{11}(t)$ as

$$p_{11}(t) = p_{11}^{(1)}(t) + p_{11}^{(2)}(t) + p_{11}^{(3)}(t) + p_{11}^{(4)}(t). \tag{37}$$

For the $p_{11}^{(1)}(t)$ and $p_{11}^{(2)}(t)$ computations refer to Figure 10. After the first kill of an A by a B, it is clear from the independence assumption and homogeneity of the marksman on each side that with equal likelihood the surviving A is either aiming at the B that killed or the B that did not kill. Thus define $p_{11}^{(1)}(t, u, v)\, du\, dv$ = probability that one of the Bs killed an A in $(t-v-dv, t-v)$, the surviving A kills the nonkilling B in $(t-u-du, t-u)$, and there is no subsequent killing until after time t; $p_{11}^{(1)}(t, u, v)\, du\, dv$ may now be written as 1/2 times the product of

$2f_B(t-v)\, dv$ = probability that one of the two Bs kills an A in the time interval $(t-v-dv, t-v)$,
$G_B(v)$ = probability that the killing B reaims and has an interkill time $>v$ (since this is the B that will survive on the B side),
$G_B(t-u)$ = probability that the nonkilling B, which will be killed by surviving A, has a time to kill $>t-u$ (this is true whether he were initially aiming at the A that was killed or not, since if he were aiming at the killed A, he merely transfers the remaining time to kill to the surviving A),
$G_A(t-v)$ = probability that killed A had a time to kill $>t-v$,
$f_A(t-u)\, du$ = probability that the surviving A kills a B in $(t-u-du, t-u)$,

and finally $G_A(u)$ = probability that the killing A reaims with an interkill time $>u$. So we have

$$p_{11}^{(1)}(t) = \int_0^t \mathrm{d}v\left[f_B(t-v)G_B(v)G_A(t-v)\int_0^v du G_B(t-u)f_A(t-u)G_A(u)\right]. \tag{38}$$

In a similar fashion we define $p_{11}^{(2)}(t, u, v)\, du\, dv$ = probability that one of the Bs killed an A in $(t-v-dv, t-v)$, the surviving A kills the killing B in $(t-u-du, t-u)$, and there is no subsequent killing until after time t, and get

$$p_{11}^{(2)}(t, u, v)\, du\, dv = f_B(t-v)\, dv\, G_B(v-u)G_B(t)G_A(t-v)f_A(t-u)\, du\, G_A(u),$$

with

$$p_{11}^{(2)}(t) = G_B(t)\int_0^t dv\left[f_B(t-v)G_A(t-v)\int_0^v du\, G_B(v-u)f_A(t-u)G_A(u)\right]. \tag{39}$$

To write expressions for $p_{11}^{(3)}(t)$ and $p_{11}^{(4)}(t)$ see Figure 11 to recall the

definitions of u and v in the transition $(2,1)\rightarrow(1,1)$. As before, we can, by interchanging A with B, u with v in Eqs. (38) and (39) set down

$$p_{11}^{(3)}(t)=\int_0^t du\left[f_A(t-u)G_A(u)G_B(t-u)\int_0^u dv\, G_A(t-v)f_B(t-v)G_A(v)\right], \tag{40}$$

and

$$p_{11}^{(4)}(t)=G_A(t)\int_0^t du\left[f_A(t-u)G_B(t-u)\int_0^u dv\, G_A(u-v)f_B(t-v)G_B(v)\right], \tag{41}$$

respectively.

4. $p_{02}(t)$, $p_{20}(t)$. We now write

$$p_{02}(t)=p_{02}^{(1)}(t)+p_{02}^{(2)}(t), \tag{42}$$

where $p_{02}^{(1)}(t)$ is the term that arises when the same B kills both As and $p_{02}^{(2)}(t)$ arises when each B kills an A. Referring to Figure 12 for the definitions of ζ and η we may write

$$p_{02}^{(1)}(t)=2\int_0^t d\zeta\left[f_B(t-\zeta)G_A(t-\zeta)\int_0^\zeta d\eta\, f_B(\zeta-\eta)G_B(t-\eta)G_A(t-\eta)\right], \tag{43}$$

and

$$p_{02}^{(2)}(t)=2\int_0^t d\zeta\left[f_B(t-\zeta)G_A(t-\zeta)\int_0^\zeta d\eta\, G_B(\zeta-\eta)f_B(t-\eta)G_A(t-\eta)\right]. \tag{44}$$

Similarly, write

$$p_{20}(t)=p_{20}^{(3)}(t)+p_{20}^{(4)}(t). \tag{45}$$

If we suppose the first B is killed in $(t-\zeta-d\zeta, t-\zeta)$ and the second B in $(t-\eta-d\eta, d\eta)$, then by simply interchanging A with B in Eqs. (34) and (35) we get

$$p_{20}^{(3)}(t)=2\int_0^t d\zeta\left[f_A(t-\zeta)G_B(t-\zeta)\int_0^\zeta d\eta\, f_A(\zeta-\eta)G_A(t-\eta)G_B(t-\eta)\right], \tag{46}$$

and

$$p_{20}^{(4)}(t)=2\int_0^t d\zeta\left[f_A(t-\zeta)\, G_B(t-\zeta)\int_0^\zeta d\eta\, G_A(\zeta-\eta)f_A(t-\eta)G_B(t-\eta)\right], \tag{47}$$

respectively.

5. $p_{01}(t), p_{10}(t)$. We write

$$p_{01}(t) = p_{01}^{(1)}(t) + p_{01}^{(2)}(t) + p_{01}^{(3)}(t) + p_{01}^{(4)}(t) \tag{48}$$

and use, as in Model 1.1 instantaneous kill rates, each of which depends on the appropriate backward recurrence time, to develop the expressions for the terms on the right-hand side of Eq. (48). Thus in the case of $p_{01}^{(1)}(t)$, which arises from the sequence $(2,2)\to(1,2)\to(1,1)\to(0,1)$, we take $p_{11}^{(1)}(t,u,v)$, in which u and v are the backward recurrence times, measured at time t, of the surviving A and B, respectively, and in the usual fashion write

$$p_{01}^{(1)}(t+\Delta) = p_{01}^{(1)}(t) + \int_0^t dv \int_0^v du\, p_{11}^{(1)}(t,u,v) r_B(v)\Delta(1 - r_A(u)\Delta).$$

After rearranging terms, dividing by Δ, letting $\Delta \to 0$, and using the initial condition $p_{01}^{(1)}(0) = 0$, we may write

$$\begin{aligned} p_{01}^{(1)}(t) = \int_0^t d\zeta \Bigg[\int_0^\zeta dv \Bigg[f_B(\zeta - v) f_B(v) G_A(\zeta - v) \\ \times \int_0^v du\, G_B(\zeta - u) f_A(\zeta - u) G_A(u)\Bigg]\Bigg]. \end{aligned} \tag{49}$$

In a similar fashion we also get

$$\begin{aligned} p_{01}^{(2)}(t) = \int_0^t d\zeta \Bigg[f_B(\zeta) \int_0^\zeta dv \Bigg[f_B(\zeta - v) G_A(\zeta - v) \\ \times \int_0^v du\, f_A(\zeta - u) G_A(u) G_B(v - u)\Bigg]\Bigg], \end{aligned} \tag{50}$$

$$\begin{aligned} p_{01}^{(3)}(t) = \int_0^t d\zeta \Bigg[\int_0^\zeta du \Bigg[f_A(\zeta - u) G_A(u) G_B(\zeta - u) \\ \times \int_0^u dv\, G_A(\zeta - v) f_B(\zeta - v) f_B(v)\Bigg]\Bigg], \end{aligned} \tag{51}$$

and

$$\begin{aligned} p_{01}^{(4)}(t) = \int_0^t d\zeta \Bigg[G_A(\zeta) \int_0^\zeta du \Bigg[f_A(\zeta - u) G_B(\zeta - u) \\ \times \int_0^u dv\, f_B(\zeta - v) f_B(v) G_A(u - v)\Bigg]\Bigg]. \end{aligned} \tag{52}$$

Again, as in Model 1.1, we may interchange A and B and u with v in the appropriate $p_{01}^{(i)}(t)$, $i = 1, 2, 3, 4$ to get each of right-hand members in

$$p_{10}(t) = p_{10}^{(1)}(t) + p_{10}^{(2)}(t) + p_{10}^{(3)}(t) + p_{10}^{(4)}(t) \tag{53}$$

as

$$p_{10}^{(1)}(t) = \int_0^t d\zeta \left[\int_0^\zeta dv \left[f_B(\zeta - v) G_B(v) G_A(\zeta - v) \times \int_0^v du G_B(\zeta - u) f_A(\zeta - u) f_A(u) \right]\right], \tag{54}$$

$$p_{10}^{(2)}(t) = \int_0^t d\zeta \left[G_B(\zeta) \int_0^\zeta dv \left[f_B(\zeta - v) G_A(\zeta - v) \times \int_0^v du\, f_A(\zeta - u) f_A(u) G_B(v - u) \right]\right], \tag{55}$$

$$p_{10}^{(3)}(t) = \int_0^t d\zeta \left[\int_0^\zeta du \left[f_A(\zeta - u) f_A(u) G_B(\zeta - u) \times \int_0^u dv\, G_A(\zeta - v) f_B(\zeta - v) G_B(v) \right]\right], \tag{56}$$

and

$$p_{10}^{(4)}(t) = \int_0^t d\zeta \left[f_A(\zeta) \int_0^\zeta du \left[f_A(\zeta - u) G_B(\zeta - u) \times \int_0^u dv\, f_B(\zeta - v) G_B(v) G_A(u - v) \right]\right]. \tag{57}$$

Model 2.1 ($a_0 = 2$, $a_f = 1$; $b_0 = 2$, $b_f = 0$; "Reselect On")

The transient states are given by

$$p_{22}(t) = (G_A(t))^2 (G_B(t))^2, \tag{58}$$

$$p_{21}(t) = G_B(t) \int_0^t d\eta\, f_A(t - \eta) G_A(t - \eta) (G_A(\eta))^2 G_B(t - \eta) + G_B(t) G_A(t) \int_0^t d\eta\, f_A(t - \eta) G_A(\eta) G_B(t - \eta). \tag{59}$$

The absorbing states are given by

$$p_{12}(t) = 2 \int_0^t d\zeta\, (G_A(\zeta))^2 G_B(\zeta) f_B(\zeta), \tag{60}$$

$$p_{11}(t) = \int_0^t d\eta \int_0^\eta du\, f_A(\eta - u) G_A(\eta - u) (G_A(u))^2 G_B(\eta - u) f_B(\eta) + \int_0^t d\eta \int_0^\eta du\, f_A(\eta - u) G_B(\eta - u) G_A(u) G_A(\eta) f_B(\eta), \tag{61}$$

$$p_{20}(t) = 2\int_0^t d\zeta \Big[f_A(t-\zeta)G_A(t-\zeta)G_B(t-\zeta)\int_0^\zeta d\eta\, f_A(\zeta-\eta)$$

$$\times\, G_A(\zeta-\eta)G_B(t-\eta)\Big]$$

$$+\int_0^t d\zeta\Big[f_A(t-\zeta)G_B(t-\zeta)\int_0^\zeta d\eta\, G_A(t-\eta)$$

$$\times\, f_A(\zeta-\eta)G_B(t-\eta)\Big]$$

$$+\int_0^t d\zeta\Big[f_A(t-\zeta)G_B(t-\zeta)\int_0^\zeta d\eta\, G_A(\zeta-\eta)$$

$$\times\, f_A(t-\eta)G_B(t-\eta)\Big]. \tag{62}$$

Model 2.2 ($a_0 = 2$, $a_t = 1$; $b_0 = 2$, $b_t = 0$; "Reselect Off")

The transient states are given by

$$p_{22}(t) = (G_A(t))^2(G_B(t))^2, \tag{63}$$

$$p_{21}(t) = 2G_A(t)G_B(t)\int_0^t d\zeta\, f_A(t-\zeta)G_A(\zeta)G_B(t-\zeta). \tag{64}$$

The absorbing states are given by

$$p_{12}(t) = 2\int_0^t (G_A(\zeta))^2 G_B(\zeta)f_B(\zeta)\, d\zeta, \tag{65}$$

$$p_{11}(t) = 2\int_0^t d\eta \int_0^\eta du\, f_A(\eta-u)G_A(u)G_A(\eta)G_B(\eta-u)f_B(\eta), \tag{66}$$

$$p_{20}(t) = 2\int_0^t d\zeta\Big[f_A(t-\zeta)G_B(t-\zeta)\int_0^\zeta d\eta\, f_A(\zeta-\eta)G_A(t-\eta)G_B(t-\eta)\Big]$$

$$+\,2\int_0^t d\zeta\Big[f_A(t-\zeta)G_B(t-\zeta)$$

$$\times\int_0^\zeta d\eta\, G_A(\zeta-\eta)f_A(t-\eta)G_B(t-\eta)\Big]. \tag{67}$$

Model 3 ($a_0 = 2$, $a_t = 1$; $b_0 = 2$, $b_t = 1$; "Reselect" not Material)

$$p_{22}(t) = (G_A(t))^2(G_B(t))^2, \tag{68}$$

while the absorbing state probabilities are

$$p_{12}(t) = 2\int_0^t (G_A(\zeta))^2 G_B(\zeta)f_B(\zeta)\, d\zeta, \tag{69}$$

$$p_{21}(t) = 2\int_0^t (G_B(\zeta))^2 G_A(\zeta) f_A(\zeta)\, d\zeta. \tag{70}$$

APPENDIX 2

As discussed in Section 4 we define parity to mean $r_A a_0^2 = r_B b_0^2$ (since we are assuming here that $a_f = b_f = 0$) so that if $a_0 = b_0$ parity is equivalent to $r_A = r_B$ and nonparity is equivalent to $r_A \neq r_B$. Table 4 shows all the gamma(2)-gamma(2) combats that we considered in the nonparity situation. Thus we have here nine cases for "reselect on" and nine cases for "reselect off" for a total of 18 different cases.

We also ran the same probabilities displayed in Table 4 for the parity case $r_A = r_B = 1$. Again there were a total 18 different cases for the hypothesized model (including "reselect on" and "reselect off"). It should be noted that in the hypothesized case strict parity obtains only when $p_A = p_B$; also, the results for, say, $p_A = 9/10$ and $p_B = 1/10$ can be obtained by interchanging the outcomes from the $p_A = 1/10$ and $p_B = 9/10$ battle. The "mixed" battles, one side gamma(2) the other side exponential, included:

1. A side gamma(2), $r_A = 1$, $p_A = 1/10$, 1/2, 9/10; B side exponential, $r_B = 1.25$; nonparity. Run with "reselect on" and "reselect off" for side A.
2. A side exponential, $r_A = 1$; B side gamma(2), $r_B = 1.25$, $p_B = 1/10$, 1/2, 9/10; nonparity. Run with "reselect on" and "reselect off" for side B.
3. A side gamma(2), $r_A = 1$, $p_A = 1/10$, 1/2, 9/10; B side exponential, $r_B = 1$; parity. Run with "reselect on" and "reselect off" for side A.
4. A side exponential, $r_A = 1$; B side gamma(2), $r_B = 1$, $p_B = 1/10$, 1/2, 9/10; parity. Run with "reselect on" and "reselect off" for side B. (Note that the results here would be obtained by just interchanging the A side and B side results in 3.)

To make the relative difference calculations discussed earlier, only two exponential–exponential and two deterministic–deterministic battles need be considered; namely, $r_A = 1.00$, $r_B = 1.25$ and $r_A = 1.00$, $r_B = 1.00$ for each pair.

Tables 5–14 which follow, present all relative differences obtained when the hypothesized model is compared with the exponential model. We have also included, for the sake of completeness, some relative difference results when the hypothesized model is compared with the deterministic model. These are shown in Tables 15 and 16. Also in Tables 7, 8, 13, and 14, where there are

Table 4. Side A, gamma(2), $r_A = 1$; side B, gamma(2), $r_B = 1.25$; nonparity. Run with both "reselect on" and "reselect off."

p_A	p_B
1/10	1/10, 1/2, 9/10
1/2	1/10, 1/2, 9/10
9/10	1/10, 1/2, 9/10

Table 5. A side gamma(2), $r_A = 1$; B side gamma(2), $r_B = 5/4$; *nonparity*; "*reselect on*."

Figure of merit	Exponential	Hypothesized model, $p_A = 1/10$					
		$p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
$E[T_D]$	0.6641	0.6884	+3.530	0.7366	+9.843	0.7757	+14.387
$\sigma[T_D]$	0.4953	0.4812	−2.930	0.4696	−5.473	0.4648	−6.565
$E[A(\infty)]$	0.6937	0.6806	−1.925	0.7498	+7.482	0.8132	+14.695
$\sigma[A(\infty)]$	0.8715	0.8635	−0.926	0.8804	+1.011	0.8919	+2.287
$E[A(\infty)]$	0.9769	0.9808	+0.398	0.8902	−9.739	0.8147	−19.909
$\sigma[B(\infty)]$	0.9034	0.8994	−0.445	0.8874	−1.803	0.8732	−3.459
$P[A]$	0.4199	0.4142	−1.376	0.4555	+7.816	0.4911	+14.498
$P[B]$	0.5798	0.5836	+0.651	0.5448	−6.424	0.5086	−13.999

Figure of merit	Exponential	*Hypothesized model*, $p_A = 1/2$					
		$p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
$E[T_D]$	0.6641	0.7272	+8.6771	0.7826	+15.1418	0.8294	+19.9301
$\sigma[T_D]$	0.4953	0.4768	−3.8800	0.4561	−8.5946	0.4491	−10.2872
$E[A(\infty)]$	0.6937	0.5738	−20.8958	0.6321	−9.7453	0.6941	+0.0576
$\sigma[A(\infty)]$	0.8715	0.8174	+6.6185	0.8363	−4.2091	0.8543	−2.0133
$E[B(\infty)]$	0.9769	1.0836	+9.8468	0.9925	+1.5718	0.9087	−7.5052
$\sigma[B(\infty)]$	0.9034	0.8925	−1.2213	0.8869	−1.8604	0.8781	−2.8812
$P[A]$	0.4199	0.3601	−16.6065	0.3970	−5.7683	0.4354	+3.5599
$P[B]$	0.5798	0.6380	+9.1223	0.6013	−3.5756	0.5646	−2.6922

Figure of merit	Exponential	Hypothesized model, $p_A = 9/10$					
		$p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
$E[T_D]$	0.6641	0.7551	+12.0514	0.8159	+18.6052	0.8682	+23.5084
$\sigma[T_D]$	0.4953	0.482	−2.6528	0.4598	−7.7207	0.4455	−11.1785
$E[A(\infty)]$	0.6937	0.4931	−40.6814	0.5389	−28.7252	0.5940	−16.7845
$\sigma[A(\infty)]$	0.8715	0.7753	−12.4081	0.7939	−9.7745	0.8141	−7.0507
$E[B(\infty)]$	0.9769	1.1715	+16.6112	1.0867	+10.1040	1.0039	+2.6895
$\sigma[B(\infty)]$	0.9034	0.8796	−2.7058	0.8802	−2.6358	0.8767	−3.0455
$P[A]$	0.4199	0.3159	−32.922	0.3477	−20.7650	0.3823	−9.8352
$P[B]$	0.5798	0.6825	+15.0476	0.6520	+11.0736	0.6167	+5.7835

Table 6. *A* side gamma(2), $r_A = 1$; *B* side gamma(2), $r_B = 5/4$; *nonparity*; "*reselect off*."

Figure of merit	Exponential	$p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
		Hypothesized model, $p_A = 1/10$					
$E[T_D]$	0.6641	0.6830	+2.7672	0.7222	+8.045	0.7552	+12.063
$\sigma[T_D]$	0.4953	0.4811	−2.9516	0.4675	−5.947	0.4590	−7.908
$E[A(\infty)]$	0.6937	0.6829	−1.5815	0.7496	+7.457	0.8112	+14.485
$\sigma[A(\infty)]$	0.8715	0.8654	−0.7049	0.8831	+1.314	0.8955	+2.680
$E[B(\infty)]$	0.9769	0.9825	−0.5700	0.9006	−8.472	0.8307	−17.600
$\sigma[B(\infty)]$	0.9034	0.9006	−0.3109	0.8916	−1.323	0.8804	−2.612
$P[A]$	0.4199	0.4145	−1.3028	0.4529	+7.286	0.4865	+13.690
$P[B]$	0.5798	0.5833	+0.6000	0.5473	−5.938	0.5132	−12.977

Figure of merit	Exponential	$p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
		Hypothesized model, $p_A = 1/2$					
$E[T_D]$	0.6641	0.7149	+7.106	0.7585	+12.446	0.7969	+16.665
$\sigma[T_D]$	0.4953	0.4737	−4.560	0.4523	−9.507	0.4419	−12.084
$E[A(\infty)]$	0.6937	0.5844	−18.703	0.6416	−8.120	0.7032	+1.351
$\sigma[A(\infty)]$	0.8715	0.8250	−5.636	0.8450	−3.136	0.8639	−0.880
$E[B(\infty)]$	0.9769	1.0823	+9.739	1.0010	+2.408	0.9238	−5.748
$\sigma[B(\infty)]$	0.9034	0.8959	−0.837	0.8928	−1.187	0.8869	−1.860
$P[A]$	0.4199	0.3636	−15.484	0.3981	−5.476	0.4344	+3.338
$P[B]$	0.5798	0.6345	+8.621	0.6004	+3.431	0.5656	−2.511

Figure of merit	Exponential	$p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
		Hypothesized model, $p_A = 9/10$					
$E[T_D]$	0.6641	0.7392	+10.160	0.7864	+15.552	0.8283	+19.824
$\sigma[T_D]$	0.4953	0.4756	−4.142	0.4522	−9.531	0.4348	−13.914
$E[A(\infty)]$	0.6937	0.5079	−36.582	0.5532	−25.398	0.6088	−13.945
$\sigma[A(\infty)]$	0.8715	0.7873	−10.695	0.8070	−7.993	0.8286	−5.177
$E[B(\infty)]$	0.9769	1.1683	+16.383	1.0942	+10.720	1.0189	+4.122
$\sigma[B(\infty)]$	0.9034	0.8847	−2.114	0.8872	−1.826	0.8867	−1.883
$P[A]$	0.4199	0.3214	−30.647	0.3509	−19.664	0.3838	−9.406
$P[B]$	0.5798	0.6771	+14.370	0.6488	+10.635	0.6154	+5.785

Table 7. A side gamma(2), $r_A = 1$; B side gamma(2), $r_B = 1$; *parity*; "*reselect on*."

Figure of merit	Exponential	$p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
		Hypothesized model, $p_A = 1/10$					
$E[T_D]$	0.7497	0.7785	+3.699	0.8281	+9.583	0.8665	+13.480
$\sigma[T_D]$	0.5568	0.5462	−1.941	0.5351	−4.055	0.5359	−3.900
$E[A(\infty)]$	0.8337	0.8278	−0.713	0.9139	+8.776	0.9919	+15.949
$\sigma[A(\infty)]$	0.8974	0.8930	−0.493	0.9002	+0.311	0.9016	+0.466
$E[B(\infty)]$	0.8337	0.8278	−0.713	0.7277	−14.566	0.6476	−28.737
$\sigma[B(\infty)]$	0.8974	0.8930	−0.493	0.8652	−3.722	0.8376	−7.139
$P[A]$	0.4997	0.4998	+0.020	0.5476	+8.747	0.5892	+15.190
$P[B]$	0.4997	0.4998	+0.020	0.4520	−10.553	0.4106	−21.700
		Hypothesized model, $p_A = 1/2$					
$E[T_D]$	0.7497	0.8281	+9.583	0.8868	+15.460	0.9326	+19.612
$\sigma[T_D]$	0.5568	0.5351	−4.055	0.5166	−7.782	0.5123	−8.686
$E[A(\infty)]$	0.8337	0.7277	−14.566	0.8078	−3.206	0.8883	+6.147
$\sigma[A(\infty)]$	0.8974	0.8652	−3.722	0.8778	−2.233	0.8859	−1.298
$E[B(\infty)]$	0.8337	0.9139	+8.776	0.8078	−3.206	0.7170	−16.276
$\sigma[B(\infty)]$	0.8974	0.9002	+0.311	0.8778	−2.233	0.8529	−5.217
$P[A]$	0.4997	0.4520	−10.553	0.4998	+0.020	0.5452	+8.346
$P[B]$	0.4997	0.5476	+8.747	0.4998	+0.020	0.4545	−9.945
		Hypothesized model, $p_A = 9/10$					
$E[T_D]$	0.7497	0.8665	+13.480	0.9326	+19.612	0.9858	+23.950
$\sigma[T_D]$	0.5568	0.5359	−3.900	0.5123	−8.686	0.5038	−10.520
$E[A(\infty)]$	0.8337	0.6476	−28.737	0.7170	−16.276	0.7931	−5.119
$\sigma[A(\infty)]$	0.8974	0.8376	−7.139	0.8529	−5.217	0.8661	−3.614
$E[B(\infty)]$	0.8337	0.9919	+15.949	0.8883	+6.147	0.7931	−5.493
$\sigma[B(\infty)]$	0.8974	0.9016	+0.466	0.8859	−1.298	0.8661	−3.614
$P[A]$	0.4997	0.4106	−21.700	0.4545	−9.945	0.4999	+0.040
$P[B]$	0.4997	0.5892	+15.190	0.5452	+8.346	0.4999	+0.040

Table 8. *A* side gamma(2), $r_A = 1$; *B* side gamma(2), $r_B = 1$; *parity*; "*reselect off.*"

Figure of merit	Exponential	$p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
		Hypothesized model, $p_A = 1/10$					
$E[T_D]$	0.7497	0.7724	+2.939	0.8129	+7.775	0.8459	+11.373
$\sigma[T_D]$	0.5568	0.5460	−1.978	0.5321	−4.642	0.5285	−5.355
$E[A(\infty Q]$	0.8337	0.8298	−0.470	0.9131	+8.696	0.9892	+15.720
$\sigma[A(\infty)]$	0.8974	0.8945	−0.324	0.9031	+0.631	0.9058	+0.927
$E[B(\infty)]$	0.8337	0.8298	−0.470	0.7384	−12.906	0.6634	−25.671
$\sigma[B(\infty)]$	0.8974	0.8945	−0.324	0.8712	−3.007	0.8473	−5.913
$P[A]$	0.4997	0.4998	+0.020	0.5445	+8.228	0.5840	+14.435
$P[B]$	0.4997	0.4998	+0.020	0.4551	−9.800	0.4158	−20.178
		Hypothesized model, $p_A = 1/2$					
Figure of merit	Exponential	$p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
$E[T_D]$	0.7497	0.8129	+7.775	0.8594	+12.765	0.8973	+16.786
$\sigma[T_D]$	0.5568	0.5321	−4.642	0.5122	−8.708	0.5039	−10.498
$E[A(\infty q]$	0.8337	0.7384	−12.906	0.8171	−2.032	0.8967	+7.026
$\sigma[A(\infty)]$	0.8974	0.8712	−3.007	0.8850	−1.401	0.8942	−0.358
$E[B(\infty)]$	0.8337	0.9131	+8.696	0.8171	−2.032	0.7322	−13.862
$\sigma[B(\infty)]$	0.8974	0.9031	+0.631	0.8850	−1.401	0.8640	−3.866
$P[A]$	0.4997	0.4551	−9.800	0.4998	+0.020	0.5430	+7.974
$P[B]$	0.4997	0.5445	+8.228	0.4998	+0.020	0.4568	−9.391
		Hypothesized model, $p_A = 9/10$					
Figure of merit	Exponential	$p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
$E[T_D]$	0.7497	0.8459	+11.373	0.8973	+16.449	0.9404	+20.279
$\sigma[T_D]$	0.5568	0.5285	−5.355	0.5039	−10.498	0.4915	−13.286
$E[A(\infty)]$	0.8337	0.6634	−25.671	0.7322	−13.862	0.8084	−3.130
$\sigma[A(\infty)]$	0.8974	0.8473	−5.913	0.8640	−3.866	0.8784	−2.163
$E[B(\infty)]$	0.8337	0.9892	+15.720	0.8967	+7.026	0.8084	−3.130
$\sigma(B(\infty)]$	0.8974	0.9058	+0.927	0.8942	−0.358	0.8784	−2.163
$P[A]$	0.4997	0.4158	−20.178	0.4568	−9.391	0.4999	+0.040
$P[B]$	0.4997	0.5840	+14.435	0.5430	+7.974	0.4999	+0.040

Table 9. *A* side gamma(2), $r_A = 1$; *B* side exponential, $r_B = 5/4$; *mixed*; *nonparity*; "*reselect on*."

Figure of merit	Exponential	Hypothesized model: $p_A = 1/10$	Relative difference (%)	$p_A = 1/2$	Relative difference (%)	$p_A = 9/10$	Relative difference (%)
$E[T_D]$	0.6641	0.6757	+1.717	0.7128	+6.832	0.7396	+10.208
$\sigma[T_D]$	0.4953	0.4918	−0.712	0.4882	−1.454	0.4938	−0.304
$E[A(\infty)]$	0.6937	0.6629	−4.646	0.5604	−23.787	0.4837	−43.593
$\sigma[A(\infty)]$	0.8715	0.8593	−1.420	0.8132	−7.169	0.7716	−12.947
$E[B(\infty)]$	0.9769	1.0044	+2.738	1.1047	+11.569	1.1899	+17.901
$\sigma[B(\infty)]$	0.9034	0.9024	−0.111	0.8940	−1.051	0.8799	−2.671
$P[A]$	0.4199	0.4052	−3.628	0.3527	−19.053	0.3102	−35.364
$P[B]$	0.5798	0.5947	+2.505	0.6471	+10.400	0.6897	+15.934

Table 10. *A* side gamma(2), $r_A = 1$; *B* side exponential, $r_B = 5/4$; *mixed*; *nonparity*; "*reselect off*."

Figure of merit	Exponential	Hypothesized model: $p_A = 1/10$	Relative difference (%)	$p_A = 1/2$	Relative difference (%)	$p_A = 9/10$	Relative difference (%)
$E[T_D]$	0.6641	0.6733	+1.366	0.7040	+5.668	0.7275	+8.715
$\sigma[T_D]$	0.4953	0.4915	−0.773	0.4846	−2.208	0.4861	−1.893
$E[A(\infty)]$	0.6937	0.6659	−4.175	0.5714	−21.404	0.4982	−39.241
$\sigma[A(\infty)]$	0.8715	0.8610	−1.220	0.8206	−6.201	0.7833	−11.260
$E[B(\infty)]$	0.9769	1.0033	+2.631	1.1005	+11.231	1.1837	+17.471
$\sigma[B(\infty)]$	0.9034	0.9031	−0.033	0.8968	−0.736	0.8846	−2.125
$P[A]$	0.4199	0.4063	−3.347	0.3570	−17.619	0.3163	−32.754
$P[B]$	0.5798	0.5936	+2.325	0.6428	+9.801	0.6836	+15.184

Table 11. *A* side exponential, $r_A = 1$; *B* side gamma(2), $r_B = 5/4$; *mixed*; *nonparity*; "*reselect on*."

Figure of merit	Exponential	Hypothesized model $p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
$E[T_D]$	0.6641	0.6768	+1.876	0.7225	+8.083	0.7598	+12.595
$\sigma[T_D]$	0.4953	0.4886	−1.371	0.4764	−3.967	0.4726	−4.803
$E[A(\infty)]$	0.6937	0.7115	+2.502	0.7794	+10.996	0.8434	+17.720
$\sigma[A(\infty)]$	0.8715	0.8757	+0.480	0.8902	+2.101	0.9000	+3.167
$E[B(\infty)]$	0.9769	0.9533	−2.476	0.8662	−12.780	0.7939	−23.051
$\sigma[B(\infty)]$	0.9034	0.9003	−0.344	0.8867	−1.883	0.8716	−3.648
$P[A]$	0.4199	0.4300	+2.349	0.4690	+10.469	0.5041	+16.703
$P[B]$	0.5798	0.5696	−1.791	0.5309	−9.211	0.4958	−16.942

Table 12. *A* side exponential, $r_A = 1$; *B* side gamma (2), $r_B = 5/4$; *mixed*; *nonparity*; "*reselect on*."

Figure of merit	Exponential	Hypothesized model $p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
$E[T_D]$	0.6641	0.6741	+1.483	0.7114	+6.649	0.7431	+10.631
$\sigma[T_D]$	0.4953	0.4885	−1.392	0.4741	−4.472	0.4663	−6.219
$E[A(\infty)]$	0.6937	0.7107	+2.392	0.7757	+10.571	0.8374	+17.160
$\sigma[A(\infty)]$	0.8715	0.8759	+0.502	0.8913	+2.221	0.9022	+3.403
$E[B(\infty)]$	0.9769	0.9560	−2.186	0.8773	−11.353	0.8106	−20.516
$\sigma[B(\infty)]$	0.9034	0.9010	−0.266	0.8905	−1.449	0.8784	−2.846
$P(A)$	0.4199	0.4291	+2.144	0.4653	+9.757	0.4984	+15.750
$P[B]$	0.5798	0.5704	−1.648	0.5345	−8.475	0.5015	−15.613

Table 13. *A* side exponential, $r_A = 1$; *B* side gamma(2), $r_B = 1$; *mixed*; *parity*; "*reselect on*."*

Figure of merit	Exponential	Hypothesized model: $p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
$E[T_D]$	0.7497	0.7636	+1.820	0.8112	+7.581	0.8481	+11.602
$\sigma[T_D]$	0.5568	0.5515	−0.961	0.5427	−2.598	0.5445	−2.259
$E[A(\infty)]$	0.8337	0.8561	+2.617	0.9407	+11.375	1.0165	+17.983
$\sigma[A(\infty)]$	0.8974	0.9000	+0.289	0.9049	+0.829	0.9045	+0.785
$E[B(\infty)]$	0.8337	0.8060	−3.437	0.7098	−17.456	0.6360	−31.085
$\sigma[B(\infty)]$	0.8974	0.8903	−0.797	0.8619	−4.119	0.8341	−7.589
$P[A]$	0.4997	0.5123	+2.459	0.5587	+8.305	0.5987	+16.536
$P[B]$	0.4997	0.4872	−2.566	0.4408	−13.362	0.4009	−24.645

*Note that setting *A* side gamma(2), $r_A = 1$ and *B* side exponential, $r_B = 1$ would be obtained by just relabeling all the *B* results as *A* results.

Table 14. *A* side exponential, $r_A = 1$; *B* side gamma(2), $r_B = 1$; *mixed*; *parity*; "*reselect off*."*

Figure of merit	Exponential	Hypothesized model: $p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
$E[T_D]$	0.7497	0.7608	+1.459	0.7999	+6.276	0.8318	+9.870
$\sigma[T_D]$	0.5568	0.5515	−0.961	0.5394	−3.226	0.5367	−3.745
$E[A(\infty)]$	0.8337	0.8551	+2.503	0.9366	+10.987	1.0104	+17.488
$\sigma[A(\infty)]$	0.8974	0.9001	+0.300	0.9068	+1.037	0.9080	+1.167
$E[B(\infty)]$	0.8337	0.8089	−3.066	0.7211	−15.615	0.6493	−28.400
$\sigma[B(\infty)]$	0.8974	0.8915	−0.662	0.8675	−3.447	0.8435	−6.390
$P[A]$	0.4997	0.5113	+2.269	0.5546	+9.899	0.5926	+15.677
$P[B]$	0.4997	0.4881	−2.377	0.4449	−12.317	0.4070	−22.776

*Note that setting *A* side gamma(2), $r_A = 1$ and B side exponential, $r_B = 1$ would be obtained by just relabeling all the *B* results as *A* results.

Table 15. *A* side gamma(2), $r_A = 1$; *B* side gamma(2), $r_B = 5/4$; *nonparity*; "*reselect on*."

Figure of merit	Deterministic	$p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
		Hypothesized model, $p_A = 1/10$					
$E[T_D]$	1.2912	0.6884	−87.565	0.7366	−75.292	0.7757	−66.456
$\sigma[T_D]$	0	0.4812	100	0.4696	100	0.4648	100
$E[A(\infty)]$	0	0.6806	100	0.7498	100	0.8132	100
$\sigma[A(\infty)]$	0	0.8635	100	0.8804	100	0.8919	100
$E[B(\infty)]$	0.8940	0.9808	+8.850	0.8902	−0.427	0.8147	−9.734
$\sigma[B(\infty)]$	0	0.8994	100	0.8874	100	0.8732	100
$P[A]$	0	0.4142	100	0.4555	100	0.4911	100
$P[B]$	1	0.5836	−71.350	0.5448	−83.553	0.5086	−96.618
		Hypothesized model, $p_A = 1/2$					
$E[T_D]$	1.2912	0.7272	−77.558	0.7826	−64.988	0.8294	−55.679
$\sigma[T_D]$	0	0.4768	100	0.4561	100	0.4491	100
$E[A(\infty)]$	0	0.5738	100	0.6321	100	0.6941	100
$\sigma[A(\infty)]$	0	0.8174	100	0.8363	100	0.8543	100
$E[B(\infty)]$	0.8940	1.0836	+17.497	0.9925	+9.924	0.9087	+1.618
$\sigma[B(\infty)]$	0	0.8925	100	0.8869	100	0.8781	100
$P[A]$	0	0.3601	100	0.3970	100	0.4354	100
$P[B]$	1	0.6380	−56.740	0.6013	−66.306	0.5646	−77.117
		Hypothesized model, $p_A = 9/10$					
$E[T_D]$	1.2912	0.7551	−70.997	0.8159	−58.255	0.8682	−48.721
$\sigma[T_D]$	0	0.4825	100	0.4598	100	0.4455	100
$E[A(\infty)]$	0	0.4931	100	0.5389	100	0.5940	100
$\sigma[A(\infty)]$	0	0.7753	100	0.7939	100	0.8141	100
$E[B(\infty)]$	0.8940	1.1715	+23.688	1.0867	+17.732	1.0039	+10.947
$\sigma[B(\infty)]$	0	0.8796	100	0.8802	100	0.8767	100
$P[A]$	0	0.3159	100	0.3477	100	0.3823	100
$P[B]$	1	0.6825	−46.520	0.6520	−53.374	0.6167	−62.153

Table 16. A side gamma(2), $r_A = 1$; B side gamma(2), $r_B = 1$; *parity*; "*reselect on*."

Figure of merit	Deterministic	Hypothesized model, $P_A = 1/10$: $p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
$E[T_D]$	∞	0.7785	$-\infty$	0.8281	$-\infty$	0.8665	$-\infty$
$\sigma[T_D]$	0	0.5462	100	0.5351	100	0.5359	100
$E[A(\infty)]$	0	0.8278	100	0.9139	100	0.9919	100
$\sigma[A(\infty)]$	0	0.8930	100	0.9002	100	0.9016	100
$E[B(\infty)]$	0	0.8278	100	0.7277	100	0.6476	100
$\sigma[B(\infty)]$	0	0.8930	100	0.8652	100	0.8376	100
$P[A]$	0	0.4998	100	0.5476	100	0.5892	100
$P[B]$	0	0.4998	100	0.4520	100	0.4106	100

Figure of merit	Deterministic	Hypothesized model, $p_A = 1/2$: $p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
$E[T_D]$	∞	0.8281	$-\infty$	0.8868	$-\infty$	0.9326	$-\infty$
$\sigma[T_D]$	0	0.5351	100	0.5166	100	0.5123	100
$E[A(\infty)]$	0	0.7277	100	0.8078	100	0.8883	100
$\sigma[A(\infty)$	0	0.8652	100	0.8778	100	0.8859	100
$E[B(\infty)]$	0	0.9139	100	0.8078	100	0.7170	100
$\sigma[B(\infty)]$	0	0.9002	100	0.8778	100	0.8529	100
$P[A]$	0	0.4520	100	0.4998	100	0.5452	100
$P[B]$	0	0.5476	100	0.4998	100	0.4545	100

Figure of merit	Deterministic	Hypothesized model, $p_A = 9/10$: $p_B = 1/10$	Relative difference (%)	$p_B = 1/2$	Relative difference (%)	$p_B = 9/10$	Relative difference (%)
$E[T_D]$	∞	0.8665	$-\infty$	0.9326	$-\infty$	0.9858	$-\infty$
$\sigma[T_D]$	0	0.5359	100	0.5123	100	0.5038	100
$E[A(\infty)]$	0	0.6476	100	0.7170	100	0.7931	100
$\sigma[A(\infty)]$	0	0.8376	100	0.8529	100	0.8661	100
$E[B(\infty)]$	0	0.9919	100	0.8833	100	0.7931	100
$\sigma[B(\infty)]$	0	0.9016	100	0.8859	100	0.4999	100
$P[A]$	0	0.4106	100	0.4545	100	0.4999	100
$P[B]$	0	0.5892	100	0.5452	100	0.4999	100

results shown for the exponential–exponential case with parity, i.e., $r_A = r_B = 1$, the win probabilities are entered as $P[A] = P[B] = 0.4997$. This is the answer obtained using the Gaussian quadrature (we know that we must have $P[A] = P[B] = 1/2$).

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